A Theory for Integer Only Numerical Analysis (Draft)

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Abstract

This draft is to be presented at the 14^{th} International Conference on $p-$adic Analysis, in Aurillac, France, July 2016. The final version of this draft is to be submitted soon afterwards. The motivation for this work, as well as a basic $1D$ version, can be found in:

Henri Alex Esbelin and Remy Malgouyres. *Sparse convolution-based digital derivatives, fast estimation for noisy signals and approximation results*, in Theoretical Computer Science **624**: 2-24 (2016).

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1 Analyzable Spaces

1.1 Dedekind-Complete Archimedean Totally Ordered Abelian Ring

This sub-section is devoted to present the rings which have enough properties to develop the following theory. Such rings turn out to be $\mathbb{Z}$ and $\mathbb{R}$. An axiomatic presentation offers however the possibility of some generalization of the theory.

Definition 1.1 Let $(A, +, ., \preceq)$ an abelian ring on which is defined a total order $\preceq$, such that

1. for $a, b, c \in A$ with $b \preceq c$ then $a + b \preceq a + c$ (translation invariance).

2. for $a, b, c \in A$ with $0_A \preceq a$ and $b \preceq c$ then $a.b \preceq a.c$ (compatibility with the product).

Such a ring is *Dedekind-complete* when any subset of A with an upper bound has a suppremum and any subset of A with a lower bound has in infemum.

It is *archimedean* when for any nonzero $l \in A$, then $A = \bigcup_{n\in\mathbb{Z}}\{a \in A \ / \ a \preceq n.l\}$ and $A = \bigcup_{n\in\mathbb{Z}}\{a \in A \ / \ n.l \preceq a\}$

Proposition 1.1 *A UNITARY Dedekind-complete archimedean totally ordered abelian ring is isomorphic (as an ordered ring) to $\mathbb{Z}$ or $\mathbb{R}$.*

Proof. Thanks to a classical result on ordered rings, it is is isomorphic (as an ordered ring) to a sub-ring of the natural ordered field $\mathbb{R}$. Up to this isomorphism, we may suppose now that A is a sub ordered ring of $\mathbb{R}$. The order is then the natural order of the real numbers.

We consider two cases.

Suppose first that $Inf(A^*_+) = 1_A$. Let a in A greater than 1. Let n_0 be $Inf\{n \in \mathbb{N}; n > a\}$. Then $0 \leq a - (n_0 - 1) < 1$ hence $0 = a - (n_0 - 1)$.

Suppose now that $A \cap (0;1) \neq \emptyset$. Then there exist in A some element b such that $0 < b < \frac{1}{2}$. Let x be a positive real number. Consider now $X = \{a \in A; 0 \leq a \leq x\}$. It is obviously bounded in A hence has a suppremum a_0 in A. We prove now that $x = a_0$.

If $x > a_0$ then for $2^n > \frac{1}{a_0 - x}$ we have $a_0 < a_0 + b^n < a_0 + \frac{1}{2^n} < x$ a contradiction.

If $x < a_0$ then for $2^n > \frac{1}{x - a_0}$ we have $x < a_0 - \frac{1}{2^n} < a_0 - b^n < a_0$ a contradiction.

□

1.2 Complete Archimedean totally Ordered Algebra

In all this paper, A, B are Dedekind-complete archimedean totally ordered abelian rings.

Definition 1.2 An *ordered algebra* on A is a tuple $(E, A, \preceq_E)$, where A is a Dedekind-complete archimedean totally ordered abelian ring, E is an $A-$algebra (with operations also denoted by $+$ and $.$) and $\preceq$ is a complete order, compatible with the order in A, that is:
if $a \in A$ and $x, y \in E$ with $x \preceq_E y$, if $0_A \preceq a$ then $ax \preceq_E ay$, and if $a \preceq_A 0_A$, then $ay \preceq_E ax$.

Definition 1.3 A *complete ordered algebra* on A is an ordered algebra $(E, A, \preceq)$, where $\preceq$ is a Dedekind-complete order.

Definition 1.4 An *archimedean ordered algebra* on A is an ordered algebra $(E, A, \preceq)$, where for any nonzero $l \in A$, then $A = \bigcup_{n \in \mathbb{Z}} \{a \in A \ / \ a \preceq n.l\}$ and $A = \bigcup_{n \in \mathbb{Z}} \{a \in A \ / \ n.l \preceq a\}$

The lexicographic order, which is of frequent use in computer sciences, does not define complete algebra on the product of complete algebras. Exemple given, let us order the cartesian product $\mathbb{R} \times \mathbb{R}$ using the lexicographic order. Let us consider the set $\left\{(1 - \frac{1}{n}, n); n \in \mathbb{N}\right\}$. It is majorized by $(1, b)$ where $b \in \mathbb{R}$, but none of this pairs are upper bounds. But also none (a, b) with $a \in \mathbb{R}$, $a < 1$ and $b \in \mathbb{R}$ is an upper bound. Hence it has non upper bound. It is no more archimedean since $\mathbb{R} \times \mathbb{R}$ is different to $\bigcup_{n \in \mathbb{Z}} \{(u, v) \in \mathbb{R} \times \mathbb{R} \ / \ (u, v) \preceq_{lex} n(0, 1)\} = \mathbb{R}_- \times \mathbb{R}$.

Example 1.1 *Let $E_1, \preceq_1$ and $E_2, \preceq_2$ be totally ordered Dedekind complete sets. If E_2 has a least element, $E_1 \times E_2$ is totally ordered Dedekind complete set for the lexicographic order.*

Indeed, let $X \subseteq E_1 \times E_2$ be a non empty subset of $E_1 \times E_2$, b_m be the least element of E_2 and (a, b) a mojorant of X.

Let us denote a_m the upper bound of $\{u_1 \in E_1; \exists u_2 \in E_2 : (u_1, u_2) \in X\}$ in E_1.

• if $a_M \in \{u_1 \in E_1; \exists u_2 \in E_2 : (u_1, u_2) \in X\}$, then let us denote b_M the upper bound of $\{u_2 \in A_1; (a_M, u_2) \in X\}$; in this case (a_M, b_M) is the upper bound of X.

• if $a_M \notin \{u_1 \in E_1; \exists u_2 \in E_2 : (u_1, u_2) \in X\}$; in this case (a_M, b_m) is the upper bound of X.

In order to enlarge the category of considered algebras, we introduce multi archimedean partial orders.

1.3 Complete Multi Archimedean partially Ordered Algebra

Definition 1.5 A *partially ordered algebra* on A is a tuple $(E, A, \preceq_E)$, where A is a Dedekind-complete archimedean totally ordered abelian ring, E is an $A-$algebra (with operations also denoted by $+$ and $.$) and $\preceq$ is a partial order, compatible with the order in A, that is:
if $a \in A$ and $x, y \in E$ with $x \preceq_E y$, if $0_A \preceq a$ then $ax \preceq_E ay$, and if $a \preceq_A 0_A$, then $ay \preceq_E ax$.

The following definitions only need $\preceq$ to be a partial order on a set E.

Definition 1.6 Let $x \in E$.

1. An element $y \in E$ is called a *close minorant of* x if and only if the order induced by $\preceq$ on the set $[y, x] = \{z \in E \ / \ y \preceq z \preceq x\}$ is a total order.
2. We say that y is a *close strict minorant of* x if, in addition, we have $x \neq y$.
3. We say that y is a *[strict] close majorant of* x if x is a [strict] close minorant of y.
4. We say that x and y are called *closely comparable* if y is either a close majorant of x or a close minorant of x.

Example 1.2 *On a cartesian product of copies of totally ordered sets, define $x \preceq y$ if and only if each coordinate x_a of x is less than the corresponding coordinate y_a of y. This order is called coordinatewise order. Then, a close strict minorant of x is a minorant of x all coordinates of which BUT ONE are equal to these of x.*

Definition 1.7 Let x and y be elements of E. An element z in E is said to be *closely between x en y* if and only if:

$$\begin{cases} \text{either} \quad x \leq z \leq y \text{ or } y \leq z \leq x \\ \text{and} \\ z \text{ is closely comparable to both } x \text{ and } y \end{cases}$$

Remark 1.1 *As usual $y \prec_E x$ is defined by $y \preceq_E x$ and $y \neq x$. It is not equivalent to x is closely strictly greater than y.*

Proof. In $\mathbb{Z}^2$ provided with the coordinatewise order, $(1, 0)$ is not closely strictly greater than $(0, 0)$ but $(0, 0) \prec (1, 1)$. □

Definition 1.8 An element x a partially ordered algebra $(E, A, \preceq_E)$ is said to be closely strictly positive if and only if for any close strict minorant y of x, there exists an element in $z \in E$, with $0_E \prec x$, which is closely between y and x (hence z is a strict close minorant of x).
x is closely strictly greater than y iff $x - y$ is closely strictly greater than 0_E.

Definition 1.9 A *multi-Archimedean partially ordered unitary algebra* is a partially ordered unitary algebra such that,

1. for any closely strictly positive element l, we have

$$E = \bigcup_{n \in \mathbb{Z}} \{e \in E \ / \ e \preceq n * l\}$$

2. Any subset of E with an upper-bound admits a unique suppremum, and any subset of E with a lower-bound admits a unique infemum.

Definition 1.10 A *complete multi-Archimedean partially ordered unitary algebra* is a multi-Archimedean partially ordered unitary algebra such that, any subset of E with an upper-bound admits a unique suppremum, and any subset of E with a lower-bound admits a unique infemum.

Furthermore, as the order $\preceq$ is translation-invariant in E, we can define the absolute value of a non zero element as follows.

Definition 1.11 Let $(E, A, \preceq_E)$ be a multi-Archimedean partially ordered unitary algebra. Let $x \in E$.

1. If x is closely comparable to 0_E, then $|x|$ is equal to x if $0_E \preceq x$, and $|x|$ is equal to $-x$ if $x \preceq 0_E$.
2. Given an element y which is closely comparable to x, the element $y - x$ is closely comparable to $x - x = 0_E$. We define

$$|x| = \sup(\{|y - x| \ / \ y \text{ is closely between } 0_E \text{ en } x \ \})$$

Example 1.3 *Let X be any set and A be a Dedekind-complete archimedean totally ordered abelian rings; then A^X the set of functions with domain X and codomain A is an ordered algebra for the order $f \preceq g$ iff for all x in X, we have $f(x) \preceq_A g(x)$. It is complete. In general it is not multi-archimedean, but it is multi-archimedean in the case $A = \mathbb{Z}$.*

Proof. $\star$ Let Y be a bouunded subset of $E = A^X$. Then there is f_0 in E such that for all f in Y and x in X, we have $f(x) \prec_A f_0(x)$. Hence the sets $\{f(x); f \in Y\}$ are all bounded, hence have an suppremum $g(x)$.
Let us prove that g is a suppremum for Y:

- for all f in Y and for all x in X, we have $f(x) \preceq_A g(x)$
- if $g' \preceq_E g$, there is some x_0 in X such that $g'(x_0) \prec_A g(x_0)$ then as the order is linear in A, there is som f in Y such that $g'(x_0) \prec_A f(x_0)$ a contradiction with the definition of $g(x_0)$.

$\star$ Let f and g be elements of $E = A^X$. Then they are comparable iff there is some x_0 in X and α in A such that $f - g = \alpha \mathbb{1}_{x_0}(x)$. Then f is closely strictly greater than 0_E iff for all x in X, we have $f(x) \prec_A 0_A$. Then in the case where $A = \mathbb{Z}$, for all x in X, we have $f(x) \prec_A 1_A$ and hence $E = \mathbb{Z}^X$ is multiarchimedean.□

1.4 Definition of an Analyzable Space

Definition 1.12 An *analysable space* on A is a tuple, $(E, A, \Omega, \mu, \preceq)$, where $(E, A, \preceq)$ is a complete multi-Archimedean partially ordered unitary algebra, Ω is a $\sigma-$algebra (or an algebra) of subsets of E, and μ is a translation-invariant $A-$valued measure on Ω, for which intervals are measurable (i.e. belongs to Ω).

Example 1.4 *In the following examples, the partial orders on the cartesian products are coordinatewise. $H : A \longrightarrow A$ are functions, called* convolution mask*, with $0_A \preceq H(a)$ for all $a \in A$. In the three following cases, $(E, A, \Omega, \mu, \preceq)$ is an analyzable space.*

- *Let $A = \mathbb{Z}$ and $E = \mathbb{Z}^d$ for some $d \in \mathbb{N}^*$, with any subset of E measurable ($\Omega = \mathcal{P}(E)$). Let us suppose that $\sum_{s\in\mathbb{A}} H(s)$ is finite.*

  We set $\mu_H(X) = \displaystyle\sum_{x\in E}\sum_{s\in A} H(s)\mathbb{1}_X(x-s) = \sum_{x\in E} \mathbb{1}_X * H(x)$, where $\mathbb{1}_X$ is the characteristic function of X.

- *Let $A = \mathbb{Z}$ and $E = \mathbb{R}^d$ for some $d \in \mathbb{N}^*$, with Ω the Borel $\sigma-$algebra. Let us suppose that $\sum_{s\in\mathbb{A}} H(s)$ is finite.*

  We set $\mu_H(X) = \displaystyle\int_E \sum_{s\in A} H(s)\mathbb{1}_X(x-s)dx = \int_E \mathbb{1}_X * H(x)dx$.

- *Let $A = \mathbb{R}$ and $E = \mathbb{R}^d$ for some $d \in \mathbb{N}^*$, with Ω the Borel $\sigma-$algebra. Let us suppose that $\displaystyle\int_A H(s)ds$ is finite.*

  We set $\mu_H(X) = \displaystyle\int_E\int_A H(s)\mathbb{1}_X(x-s)dsdx = \int_E \mathbb{1}_X * H(x)dx$, where integrals represent usual Lebesgues integral, with respect to the usual Lebesgues measure.

1.5 Integrals

Definition 1.13 Let $(E, A, \Omega_E, \mu, \preceq_E)$ be an analysable space and $(F, A, \preceq_F)$ be a multi-Archimedean partially ordered unitary algebra on the same Dedekind-complete archimedean totally ordered abelian ring A. Let f be a function with domain E and codomain $F_+ = \{y \in F; 0 \preceq_F y\}$.
When the set

$$\left\{\sum_{i\in I} y_i\mu(E_i); I \text{ finite}, \{E_i; i \in I\} \text{ partition of } E, E_i \in \Omega_E, y_i \in F, \forall x \in E_i, y_i \preceq_F f(x)\right\}$$

is bounded, f integrable. Let us denote $\int_E f(x)d\mu(x)$ the upper bound in A of this set.
Let f be a function with domain E and codomain F. Let f_+ be define by $f_+(x) = f(x)$ if $0_F \preceq_F f(x)$ and $f_+(x) = 0_F$ otherwise and $f_- = f_+ - f$.
f is integrable iff f_+ and f_- are integrable and in this case

$$\int_E f(x)d\mu(x) = \int_E f_+(x)d\mu(x) - \int_E f_-(x)d\mu(x)$$

Proposition 1.2 *If f is integrable and $a \in A$, then $a.f$ is integrable and $\int_E a.f(x)d\mu(x) = a\int_E f(x)d\mu(x)$.*

Proof. If $a \in A$ and $0_A \preceq_A a$, then $\{x; x \in X\}$ and $\{a.x; x \in X\}$ are simultaneously bounded or unbouded and if bounded, $a.Sup\{x; x \in X\} = Sup\{a.x; x \in X\}$, etc. □

Proposition 1.3 *If $\mu(\{x \in E; f(x) \neq 0_F\}) = 0_A$, and f is integrable, then $\int_E f(x)d\mu(x) = 0_F$.*
If $\mu(\{x \in E; f_1(x) \neq f_2(x)\}) = 0_A$, and f_1 and f_2 are integrable, then $\int_E f_1(x)d\mu(x) = \int_E f_2(x)d\mu(x)$.

Proof. □

Proposition 1.4 *If f is A_+valued and integrable, and for all x in E, we have $0_A \preceq f(x)$ and $\int_E f(x)d\mu(x) = 0_A$, then $\mu(\{x \in E; f(x) \neq 0_A\}) = 0_A$.*

Proof. This is the case because $A = \mathbb{R}$ or $A = \mathbb{Z}$. In the first case $\{x \in A; 0_A \prec f(x)\} = \{x \in A; 1 \preceq f(x)\}$and in the second case $\{x \in A; 0_A \prec f(x)\} = \cup_{i\in\mathbb{N}}\{x \in A; \frac{1}{n} \preceq f(x)\}$. □

Proposition 1.5 *Let $(f_1, ..., f_d)$ be a function with domain $(E, A, \Omega_E, \mu, \preceq_E)$ an analysable space and with codomain $\prod_{i=1}^{i=d}(F_i, A_i, \preceq_{F_i})$ a multi-Archimedean partially ordered unitary algebra defined form the multi-Archimedean partially ordered unitary algebras F_i by the composantewise order and the composantewise multiplication on the same Dedekind-complete archimedean totally ordered abelian ring A.*

Then $(f_1, ..., f_d)$ is integrable iff every f_i are integrable.

Proof. □

Proposition 1.6 *(Affine change of variable in a double integral) Let $(E, A, \Omega_E, \mu, \preceq_E)$ be an analysable space and $(F, A, \preceq_F)$ be a multi-Archimedean partially ordered unitary algebra on the same Dedekind-complete archimedean totally ordered abelian ring A. Let f be a function with domain E and codomain F.*
cartesian product $E\times E$, seen as an analyzable space for the product measure and coordinatewise order and operations. Let $T : E \times E \longmapsto E \times E$ be an affine transformation of the form:

$$T(x, y) = (a_{0,0}x + a_{0,1}y, a_{1,0}x + a_{1,1}y)$$

with $a_{0,0}, a_{0,1}, a_{1,0}, a_{1,1} \in A$. We denote $Det(T) = a_{0,0}.a_{1,1} - a_{1,0}.a_{0,1}$, which we assume to be inversible in A. Then for $X \in E \times E$:

$$\int_X f(x)dx = \int_{T^{-1}(X)} Det(T).f \circ T(u)du$$

Proof. □

Proposition 1.7 *(Fubini-Tonelli theorem) Let $(E_1, A, \Omega_{E_1}, \mu_1, \preceq_1)$ and $(E_2, A, \Omega_{E_2}, \mu_2, \preceq_2)$ be analysable spaces and $(F, A, \preceq_F)$ be a multi-Archimedean partially ordered unitary algebra on the same ring A.*
Let $f : E_1 \times E_2 \longmapsto F_+$ be an integrable function over the cartesian product $E_1 \times E_2$, seen as an analyzable space for the product measure and composantewise order.
Then $x \to \int_{E_2} f(x, y)d\mu_2(y)$ is mesurable sur E_1, and we have

$$\int_{E_1\times E_2} f(x, y)d(\mu_1 \otimes \mu_2) = \int_{E_1} \left(\int_{E_2} f(x, y)d\mu_2(y) \right) d\mu_1(x)$$

Proof. $\star$ First case: $f = \chi_E$ avec $E \in \Omega_{E_1} \otimes \Omega_{E_2}$. Then for a fixed $x \in E_1$, we have $\chi_E(x, y) = 1$ iff $y \in \{y \in E_2, (x, y) \in E\}$ (which we denote E_x) is mesurable. Then $\int_{E_2} \chi_E(x, y) d\mu_2(y) = \int_{E_x} d\mu_2(y) = \mu_2(E_x)$ and $\int_{E_1 \times E_2} \chi_E(x, y) d(\mu_1 \otimes \mu_2) = \mu_1 \otimes \mu_2(E) = \int_{E_2} \mu_1(E_y) d\mu_2(y)$ from the definition.

$\star$ By linearity, this remains true for positive fonctions $\sum_{i \in I} y_i \mathbb{1}_{E_i}$, where I is finite,$\{E_i; i \in I\}$ is a partition of $E_1 \times E_2$ and $E_i \in \Omega_{E_1} \otimes \Omega_{E_2}$.

$\star$ It remains true for mesurable positive functions on $(E_1 \times E_2, \Omega_{E_1} \otimes \Omega_{E_2})$. $\square$

1.6 Convolutions In Analyzable Spaces

Definition 1.14 Let $(E, A, \Omega, \mu, \preceq)$ be an analyzable space over an ring A. Let $l_1, l_2 \in E \cup \{-\infty, +\infty\}$. The *Interval of E between l_1 and l_2*, denoted by $[l_1, l_2]_E$ (or simply $[l_1, l_2]$ for short if no confusion can arise), is defined by

$$[l_1, l_2]_E = \{x \in E \ / \ l_1 \preceq x \preceq l_2\}$$

We define similarly open or semi-open bounded or unbounded intervals using the classical strict order.

Remark 1.2 *In $\mathbb{Z}^2$ we have $](0,0),(1,1)] = \{(1,0),(0,1),(1,1)\}$, however $(1,0)$ and $(0,1)$ are not closely strictly greater than $(0,0)$.*

Lemma 1.1 *Let $(E, A, \Omega_E, \mu, \preceq_E)$ be an analysable space.*
Let $l \in E$ such that l is closely strictly greater than $(0,0)$. Then we have the following partition of E:

$$E = \bigcup_{s \in \mathbb{Z}} [sl, (s+1)l[_E$$

Proof. $\square$

Notation 1.1 *Let $(E, A, \Omega_E, \mu, \preceq_E)$ be an analysable space and $(F, A, \preceq_F)$ be a multi-Archimedean partially ordered unitary algebra on the same Dedekind-complete archimedean totally ordered abelian ring A. Let f be a function with domain E and codomain F.*
Let $I = [l_1, l_2[$ be a (possibly unbounded) interval in E. We denote

$$\int_{l_1}^{l_2} f(x) dx = \int_{[l_1, l_2[} f(x) dx$$

Definition 1.15 Let $(E, \Omega, \mu, \preceq)$ and $(F, \Omega_F, \mu_F, \preceq)$ be analyzable spaces over an ordered abelian ring A. Let $K : E \longrightarrow F$ (or $K : E \longrightarrow A$, which can be identied to the $F-$valued function $a \longmapsto K(a).1_F$) be an integrable function, and $f : E \longrightarrow F$ be an analyzable function. We define the *convolution product* $K * f : E \longmapsto F$ of f by K by setting for $x \in E$:

$$K * f(x) = \int_E f(u) K(x - u) du$$

In the following definition, we consider a mesured space similar to Example 1.4, in a general setting.

Definition 1.16 Let $(E, \Omega, \mu, \preceq)$ be an analyzable space over an ordered abelian ring A. Let $H : A \longrightarrow A$ be a map such that $0_A \preceq H(a)$ for all $a \in A$ and the integral $\displaystyle\int_A H(a)da$ is finite. We consider a new measure on E defined by

$$\text{For } X \in \Omega,\ \mu_H(X) = \int_X \int_E H(u)\mathbb{1}_X(x-u)dudx = \int_X \mathbb{1}_X * H(x)dx$$

where $\mathbb{1}_X$ is the characteristic function of X, which to $x \in E$ associates 1_A if $x \in X$ and 0_A if $x \notin X$.

Then $(E, \Omega, \mu_H, \preceq)$ is an analyzable space. This analyzable space $(E, \Omega, \mu_H, \preceq)$ is called the *convolved analyzable space associated to* $(E, \Omega, \mu, \preceq)$ *with convolution kernel* H.

Remark 1.3 *Let* $(E, \Omega, \mu_H, \preceq)$ *be the convolved analyzable space associated to* $(E, \Omega, \mu, \preceq)$ *with a convolution kernel* H. *Then, for any* $(F, A, \preceq_F)$ *multi-Archimedean partially ordered unitary algebra on* A *and any* $\mu-$*integrable function* $f : E \longrightarrow F$,

$$\int_X f d\mu_H = \int_X (H * f)d\mu$$

1.7 Normed Analyzable Spaces and Functional Norms

Definition 1.17 We call a *Norm* over an analyzable space M over a ring A a function $N : M \longrightarrow A$ with the following properties:

1. $N(x) \succeq 0_A$ for any $x \in M$;
2. $N(x+y) \preceq N(x) + N(y)$ for all $x, y \in M$;
3. $N(a.x) = abs(a)N(x)$
4. If $N(x) = 0$, then $x = 0_M$.

We often denote by $||x||$ the norm of an element $x \in M$, instead of a notation of the form $N(x)$, in which case the norm itself is denoted by $||.||$.

Definition 1.18 We call an *Analyzable Space Norm* over an analyzable space $(E, A, \Omega, \mu, \preceq_E)$ over an ordered abelian ring A a norm $||.||$ which is compatible with the order on E, that is, a norm for which if $0_E \preceq_E x \preceq_E y$ in E then $0_A \preceq_A ||x|| \preceq_A ||y||$ in A.

Remark 1.4 *If a subset of* E *is bounded for the order, then by definition it is bounded for the norm. The converse is true.*

Proof. Let us suppose that a in A is such that for all x in X, we have $a \preceq_A ||x||$. From archimedean property in A, there exist n such that $||a|| \prec_A n.||1_E||$. Let us suppose that X is not bounded for the order. Then it exists in X some x such that $n.1_E \preceq x$, hence $||a|| \prec_A n.||(1_E)|| \preceq ||x||$ a contradiction. □

Definition 1.19 Let $(E, A, \Omega, \mu, \preceq)$ and $(F, A, \Omega_F, \mu_F, \preceq)$ be analyzable spaces over a A. Let $||.||$ be a norm on F. Let $f : E \longrightarrow F$ be a measurable function. We say that f has finite $1-$norm on a measurable subset $X \subset E$ if the following integral exists and is finite:

$$||f||_1 = \int_X ||f(x)||dx$$

This integral is then called the $1-$*norm of* f on X, or simply the $1-$norm of f if $X = E$.

Definition 1.20 Let $(E, A, \Omega, \mu, \preceq)$ and $(F, A, \Omega_F, \mu_F, \preceq)$ be analyzable spaces over a ring A. Let $||.||$ be a norm on F. Let $f : E \longrightarrow F$ be a measurable function. We say that f has finite $\infty-$norm on a measurable subset $X \subset E$ if $x \longmapsto ||f(x)||$ has an upper bound on X. We then denote $||f||_\infty = \sup_X ||f(x)||$.

Notation 1.2 *Let $(E, \Omega, \mu, \preceq)$ and $(F, A, \Omega_F, \mu_F, \preceq)$ be analyzable spaces over a ring A. Let $||.||$ be a norm on F. Let $\alpha \in \mathbb{N}^* \cup \{\infty\}$. We denote by $\mathbf{F}_\alpha(E, F, \mu)$ the space of all measurable functions from E to F with finite $\alpha-$norm. This space is naturally provided with the norm $||.||_\alpha$. For convenience, we denote $\mathbf{F}_0(E, F, \mu)$ the space of all measurable functions from E to F, which is not naturally a normed algebra.*

1.8 Differentiation and Averaging Operators on Analyzable Spaces

In this section, we consider $(E, A, \Omega, \mu, \preceq)$ and $(F, \Omega_F, \mu_F, \preceq)$ two analyzable spaces over the same ordered abelian ring A.

Definition 1.21 (Integral Based Primitive Operator) Let $f \in \mathbf{F}_1(E, F, \mu)$. We define the *integral based primitive of f* $\mathcal{I}_\mu(f) : E \mapsto F$, by

$$(\mathcal{I}_\mu(f))(x) = \int_0^x f(u)du$$

Definition 1.22 Let $\Phi : \mathbf{F}_1(E, F, \mu) \longrightarrow \mathbf{F}_1(E, F, \mu)$ be a linear operator. We say that Φ *commutes with the integral based primitive operator* if for any $f \in \mathbf{F}_1(E, F, \mu)$, we have

$$\mathcal{I}_\mu(\Phi(f)) = \Phi(\mathcal{I}_\mu(f))$$

.

Example 1.5 *Due to Lemma 1.3 a convolution operator, of the form $f \longmapsto H * f$ for some convolution kernel $H \in \mathbf{F}_1(E, F, \mu)$, commutes with the primitive operator.*

Definition 1.23 (Differentiation Operator) Let $\Delta : \mathbf{F}_1(E, F, \mu) \longmapsto \mathbf{F}_1(E, F, \mu)$ be a linear operator. We say that Δ is a *differentation operator* if the two following conditions are satisfied:

1. $\Delta(f)$ commutes with the integral based primitive operator.
2. for any $f \in \mathbf{F}_1(E, F, \mu)$ the function $\Delta(\mathcal{I}_\mu(f))$ exists and is equal to f.

Proposition 1.8 *Let $\Delta : \mathbf{F}_1(E, F, \mu) \longmapsto \mathbf{F}_1(E, F, \mu)$ be a differentiation operator. Then $\Delta(f)$ is the zero function if f is constant on E.*

Proof. For $f \in \mathbf{F}_1(E, F, \mu)$, we have $\mathcal{I}_\mu(\Delta(f)) = \Delta(\mathcal{I}_\mu(f)) = f$. So, if f is constant, $\mathcal{I}_\mu(\Delta(f))$ is constant, which implies that $||\Delta(f)||_1 = 0$ and $\Delta(f)$ is zero almost anywhere. □

Lemma 1.2 *Let $\Phi_1, \Phi_2 : \mathbf{F}_1(E, F, \mu) \longmapsto \mathbf{F}_1(E, F, \mu)$ be two linear operators which commute with the integral based primitive operator and coincide on functions for the form $\mathcal{I}_\mu(f)$ for $x \in E$. Then Φ_1 and Φ_2 are equal on $\mathbf{F}_1(E, F, \mu)$.*

Proof. Let us consider the linear operator $N(f) = \Phi_1(f) - \Phi_2(f)$. Then, for any function of the form $F = \mathcal{I}_\mu(f)$, we have $N(f) = 0$ allmost anywhere. Let $I = [l_1, l_2[$ be an interval in E. Suppose that N is non zero. Then there exists a function $F \in \mathbf{F}_1(E, F, \mu)$ such that $N(F) \neq 0$ on a non zero measure set.

Since $N(F)$ is not almost anywhere zero, we have $0_F \neq \mathcal{I}_\mu(N(F)) = N(\mathcal{I}_\mu(F))$, which contradicts the definition of N, together with hypothesis on Φ_1 and Φ_2. □

As a direct application of Lemma 1.2, we obtain:

Proposition 1.9 *(Uniqueness of The Differentiation Operator) If Δ_1 and Δ_2 are two differentiation operators on $\mathbf{F}_1(E, F, \mu)$, then they are equal.*

Proposition 1.10 (Translation Invariance The Differentiation Operator) *If Δ is a differentiation operators on $\mathbf{F}_1(E, F, \mu)$, then they Δ is translation invariant, that is: if $g(x) = f(x + x_0)$ for all $x \in E$ and for some $x_0 \in E$, then $(\Delta(g))(x) = (\Delta(f))(x + x_0)$ for all $x \in E$.*

Proof. Due to Lemma 1.2, is is sufficient to prove that the linear operator $\Delta(\mathcal{I}_\mu(f))(x) - \Delta(\mathcal{I}_\mu(f)(x + x_0)) = 0_F$ for any $x \in E$. for any $f \in \mathbf{F}_1(E, F, \mu)$. But

$$\Delta(\mathcal{I}_\mu(f))(x) - \Delta(\mathcal{I}_\mu(f))(x + x_0) = \Delta(K) = 0$$

where K is the constant function with $K(x) = \displaystyle\int_0^{x_0} f(u)du$. □

Lemma 1.3 *Let $f, H \in \mathbf{F}_1(E, F, \mu)$. Then $\mathcal{I}_\mu(H * f) = H * \mathcal{I}_\mu(f)$*

Proof.

$$\begin{aligned}
\mathcal{I}_\mu(H * f) = \int_0^x H * f(u)du &= \int_0^x \int_E H(v)f(u - v)dvdu = \int_E H(v) \int_0^x f(u - v)dudv \\
&= \int_E H(v) \int_{-v}^{x-v} f(u)dudv = \int_E H(v)((\mathcal{I}_\mu(f))(x - v) - (\mathcal{I}_\mu(f))(-v))dv \\
&= H * (\mathcal{I}_\mu(f))(x) - H * (\mathcal{I}_\mu(f))(0) = H * (\mathcal{I}_\mu(f))(x)
\end{aligned}$$

□

Lemma 1.4 *Let $f, H \in \mathbf{F}_1(E, F, \mu)$ and let Δ be a differentiation operator on $\mathbf{F}_1(E, F, \mu)$. Then,*

$$\Delta(H * f) = H * \Delta(f) = \Delta(H) * f$$

Proof. By applying Lemma 1.2 with $\Phi_1(f) = \Delta(H * f)$ and $\Phi_2(f) = H * \delta(f)$, using Lemma 1.3 to establish the hypothesis of Lemma 1.2, we obtain that $\Delta(H * f) = H * \Delta(f)$. By changing the roles of H and f, we obtain similarly that $\Delta(H * f) = \Delta(H) * f$. □

Definition 1.24 (Averaging Operator) Let $\Lambda : \mathbf{F}_1(E, F, \mu) \longrightarrow \mathbf{F}_1(E, F, \mu)$ be a linear operator which commutes with the primitive operator. We say that Λ is an *averaging operator* if $\Lambda(\mathbb{1}_{E,F}) = \mathbb{1}_{E,F}$.

Example 1.6 *Due to Lemma 1.3 a convolution operator, of the form $f \longmapsto H * f$ for some convolution kernel $H \in \mathbf{F}_1(E, F, \mu)$, with $\int_E H(u)du = 1_F$, is an averaging operator.*

Definition 1.25 (Averaged Differentiation Operator) Let $\Delta : \mathbf{F}_1(E, F, \mu) \longrightarrow \mathbf{F}_1(E, F, \mu)$ be a linear operator. We say that Δ is an *averaged differentiation operator* if and only if both following conditions are satisfied:

1. Δ is zero on any constant function.
2. We consider the linear operator $\Lambda_\Delta \mathbf{F}_1(E, F, \mu) \longrightarrow \mathbf{F}_1(E, F, \mu)$ defined by $\Lambda_\Delta(f) = \Delta(\mathcal{I}_\mu(f))$. Then the operator Λ_Δ is an averaging operator (Namely, $\Delta(\mathcal{I}_\mu(\mathbb{1}_{E,F})) = \mathbb{1}_{E,F}$).

Example 1.7 *Let Δ be a differentiation operator, or an averaged differentation operator on $\mathbf{F}_1(E, F, \mu)$. For $f \in \mathbf{F}_1(E, F, \mu)$, let us set $\Delta_H(f) = \Delta(H * f)$. Then the operator Δ_H is an averaged differentiation operator on $\mathbf{F}_1(E, F, \mu_H)$.*

More generally, the composition of an averaging operator with a differentiation operator is an averaged differentation operator, and conversely.

Example 1.8 *Let Λ be an averaging operator on $\mathbf{F}_1(E, F, \mu)$. For $f \in \mathbf{F}_1(E, F, \mu)$ and $l \in E$ with $l \succ 0_E$, let us set $(\Delta(f))(x) = (\Lambda(f))(x + 1_E) - (\Lambda(f))(x)$. Then the operator Δ is an averaged differentiation operator on $\mathbf{F}_1(E, F, \mu_H)$.*

Proposition 1.11 *(Translation Invariance an Averaged Differentiation Operator) If Δ is an averaging differentiation operators on $\mathbf{F}_1(E, F, \mu)$, then they Δ is translation invariant, that is: if $g(x) = f(x + x_0)$ for all $x \in E$ and for some $x_0 \in E$, then $(\Delta(g))(x) = (\Delta(f))(x + x_0)$ for all $x \in E$.*

Proof. Due to Lemma 1.2, is is sufficient to prove that the linear operator $\Delta(\mathcal{I}_\mu(f))(x) - \Delta(\mathcal{I}_\mu(f)(x + x_0)) = 0_F$ for any $x \in E$. for any $f \in \mathbf{F}_1(E, F, \mu)$. But

$$\Delta(\mathcal{I}_\mu(f))(x) - \Delta(\mathcal{I}_\mu(f))(x + x_0) = \Delta(K) = 0$$

where K is the constant function with $K(x) = \int_0^{x_0} f(u)du$. □

1.9 Fixed Denominator Rational Analyzable Space

In this section, we consider $(E, \Omega, \mu, \preceq)$ an analyzable space over a ring A.

Let $l \in A$, with $l \succ 0_E$. For convenience, we also denote $l = l * 1_E$ the element of E multiple of the unit element in E. For $X \subset E$, we consider the set

$$X/l = \{\frac{x}{l} \ / \ x \in X\}$$

Conversely, for $Y \subset E/l$, we define $lY = \{x \in E \ / \ \frac{x}{l} \in Y\}$. The set E/l is naturally in one to one correspondance with E through the map $x \longmapsto \frac{x}{l}$. The inverse map is the map which to some $y = \frac{x}{l} \in E/l$ associates $ly \stackrel{\text{def}}{=} x$.

We can be naturally provide E/l with an analyzable space structure $(E/l, \Omega_l, \mu_l, \preceq_l)$, which is isomrophic to $(E, \Omega, \mu, \preceq)$, by setting:

1. For $Y \subset E/l$, the set Y is in Ω_l if and only if $Y = X/l$ for some $X \in \Omega$. In other words, $lY \in Omega$.

2. We set $\mu_l(Y) = \mu(lY)$ for $Y \in \Omega_l$.

3. $y_1 \preceq y_2$ if and only if $ly_1 \preceq ly_2$.

Note that in the case when A is a field and $E = A$ (for examle $A = E = \mathbb{R}$), the E/l can be seen as E itself, and the natural isomorphism from E to E/l can be seen as an automorphism.

2 Digital Differentiation

In the sequel of this section, we consider the following structures. Let $\mathcal{R}$ be either the ring $\mathbb{Z}$ or the ring $\mathbb{R}$. Let $d \in \mathbb{N}$ and, for $a = 1, \dots, d$, let $\mathcal{A}_a$ be an analyzable space over $\mathcal{R}$. Let $d' \in \mathbb{N}$, and for $a = 1, \dots, d'$, let $\mathcal{A'}_a$ be an analyzable space over $\mathcal{R}$. We denote

$$\mathcal{M} = \prod_{a=1}^{d} \mathcal{A}_a \text{ and } \mathcal{M}' = \prod_{a=1}^{d'} \mathcal{A'}_a$$

For $a \in \{1, \dots, d\}$, we consider e_a the element of $\mathcal{M}$, the i^{th} coordinate of which is $1_{\mathcal{A}_i}$ if $i = a$, and $0_{\mathcal{A}_i}$ otherwise. We denote by $\mathcal{Z}_d$ the sub-algebra of $\mathcal{M}$ generated by the e_a's, for $a = 1, \dots, d$. Similarly, for $a \in \{1, \dots, d'\}$, we consider f_a the element of $\mathcal{M}'$, the i^{th} coordinate of which is $1_{\mathcal{A'}_a}$ if $i = a$, and $0_{\mathcal{A'}_a}$ otherwise. We denote by $\mathcal{Z'}_{d'}$ the sub-algebra of $\mathcal{M}'$ generated by the f_a's, for $a = 1, \dots, d'$.

2.1 Rapidly Decreasing and Moderately Increasing Multi-sequences

Definition 2.1 Let $\mathbf{u}$ be a multi-sequence in $\mathcal{M'}^{\mathcal{Z}_d}$. We say that $\mathbf{u}$ is *rapidly decreasing* if and only if for any polynomial function π on $\mathcal{Z}_d$, the function $I \longmapsto \pi(I)u(I)$ is bounded. We denote by $\mathcal{D}[\mathcal{Z}_d, \mathcal{M}']$ the set of rapidly decreasing multi-sequences in $\mathcal{M'}^{\mathcal{Z}_d}$.

Remark 2.1 *The space $\mathcal{D}[\mathcal{Z}_d, \mathcal{M}']$ of rapidly decreasing multi-sequences is stable under inner addition, inner multiplication, and stable under multiplication by a polynomial function.*

Lemma 2.1 *Let $\mathbf{u}$ a rapidly decreasing multi-sequence and let π be an $\mathcal{M}'-$valued polynomial function on $\mathcal{M}$. For $I \in \mathcal{Z}_{d-1}$ and $i \in \mathcal{A}_d$, let us denote by $u(I, i)$ [resp. $\pi(I, i)$] the image under $\mathbf{u}$ [resp. under π] of the concatenation of I and (i). Then the multi-sequence defined on $\mathcal{Z}_{d-1}$ by*

$$s_d(I) = \sum_{i \in \mathcal{A}_d} |u(I, i)||\pi(I, i)|$$

is well defined and bounded on $\mathcal{Z}_{d-1}$.

Proof. It is sufficient to prove this property when π is a monomial and, due to Remark 2.1, it is sufficient to prove it for polynomials of degree 0. In other words, we just need to show that the sum of the values of the multi-sequence $\mathbf{u}$ itself is absolutely convergent.

First we prove that for $d \geq 1$, the sum:

$$s_d(I) = \sum_{i \in \mathcal{A}_d} u(I, i)$$

is well defined for $I \in \mathcal{Z}_{d-1}$, and that the multi-sequence $\mathbf{s_d}$ itself is rapidly decreasing on $\mathcal{Z}_{d-1}$. Since $\mathbf{u}$ is rapidly decreasing, we can find $K > 0_{\mathcal{M}'}$ such that for $I \in \mathcal{Z}_{d-1}$ and $i \in \mathcal{A}_d$, we have $u(I,i) \leq K$ and $i^2 u(I,i) \leq K$. For $N \in \mathbb{N}^*$, we have

$$\begin{array}{rcl}(N!)^2 \sum_{i=1}^{N} |u(I,i)| & \leq & (N!)^2 |u(I,1)| + \sum_{i=2}^{N} i^2 |u(I,i)| * \frac{(N!)^2 1_{\mathcal{M}'}}{i^2} \\ & \leq & (N!)\,(K + K * 2)\end{array}$$

Note that the expression $\frac{N!}{i^2} 1_{\mathcal{M}'}$ denotes a well defined element of the algebra $\mathcal{M}'$ over the ring $\mathcal{R}$. Indeed, by expanding the expression of $(N!)^2$ and simplifying by i^2 to get an integer value, which is then multiplied by $1_{\mathcal{M}'}$ in the algebra $\mathcal{M}'$.

Hence, we get $\sum_{i \in \mathbb{N}^*} |u(I,i)|$ is well defined and bounded by $3K$. By a similar argument for $i < 0$, we get that $\sum_{i \in \mathcal{A}_d} |u(I,i)|$ is well defined and bounded on $\mathcal{Z}_{d-1}$. □

Lemma 2.2 *Let us consider the multi-sequence* $\mathbf{v}$ *defined on* $\mathcal{Z}_{d-1}$ *by* $v(I) = \sum_{i \in \mathcal{A}_d} u(I,i)$, *which is well-defined due to from Lemma 2.1, Then,* $\mathbf{v}$ *is a rapidly decreasing multi-sequence.*

Proof. Let π be a polynomial function on $\mathcal{Z}_{d-1}$. Then, by considering π as a function on $\mathcal{Z}_d$ (which does not depend on the d^{th} coordinate), we get by Remark 2.1 that the multi-sequence $I \longmapsto \pi(I)u(I)$ is rapidly decreasing. From Lemma 2.1, we get that the multi-sequence $I \longmapsto \pi(I)v(I)$ on $\mathcal{Z}_{d-1}$ is bounded, which proves that $\mathbf{v}$ is rapidly decreasing. □

Lemma 2.3 *For any rapidly decreasing multi-sequence* $\mathbf{u}$ *and any polynomial* π *on* $\mathcal{Z}_d$, *the following series is absolutely convergent:*

$$\sum_{I \in \mathcal{Z}_d} \pi(I)u(I)$$

In particular, the multi-sequence $\mathbf{u}\pi$ *is bounded.*

The proof follows immediately by induction using Lemma 2.1 and Lemma 2.2.

Proposition 2.1 *For any rapidly decreasing multi-sequence* $\mathbf{u}$ *and any polynomial* π *on* $\mathcal{Z}_d$, *the multi-sequence* $\mathbf{u}\pi$ *is rapidly decreasing.*

Definition 2.2 Let $\mathcal{I}$ be a sub-algebra of $\mathcal{M}$ containing $\mathcal{Z}_d$ (typically, $\mathcal{I} = \mathcal{Z}_d$ or $\mathcal{I} = \mathcal{M}$). Let $\mathbf{u}$ be a function in ${\mathcal{M}'}^{\mathcal{I}}$. We say that $\mathbf{u}$ is *moderately increasing* if and only if there exist a bounded subset B of $\mathcal{I}$ and a polynomial π on $\mathcal{I}$ such that for any $I \in \mathcal{I} \backslash B$ we have $|u(I)| \leq |\pi(I)|$. We denote by $\mathcal{P}[\mathcal{I}, \mathcal{M}']$ the set of moderately increasing multi-sequences in ${\mathcal{M}'}^{\mathcal{I}}$.

Remark 2.2 *The product of a rapidly decreasing multi-sequence by a moderately increasing multi-sequence is rapidly decreasing.*

Remark 2.3 *The space* $\mathcal{P}[\mathcal{I}, \mathcal{M}']$ *of* $\mathcal{M}'-$*valued moderately increasing multi-sequences over a sub-algebra* $\mathcal{I}$ *of* $\mathcal{M}$ *is stable under inner addition, inner multiplication, and multiplication by a polynomial.*

2.2 Digital Differentiation, Tensor Products

First, we introduce a few notations about multi-indices.

Notation 2.1 *Let $\mathcal{P} = \prod_{a=1}^{d} \mathcal{B}_a$ be a Cartesian product of d analyzable spaces (e.g. the Cartesian product $\mathcal{P}$ can be $\mathbb{Z}^d$ or $\mathcal{Z}_d$ over the ring $\mathbb{Z}$, or possibly $\mathcal{M}$ or an ideal $\mathcal{I}$ over the ring $\mathcal{R}$). Let $(I(a))_{a=1,\dots,d} \in \mathcal{P}$ be a multi-index. We shall use the following notations:*

1. *For $a \in \{1,\dots,d\}$ and for $j \in \mathbb{Z}$ or $j \in \mathcal{R}$ or $j \in \mathcal{B}_a$, we denote by $L(a,j)$ the element in $\mathcal{P}$, all coordinates of which are zero, except the a's coordinate which is equal to $j.1_{\mathcal{B}_a}$.*
2. *For $v \in \mathcal{P}$, for $a \in \{1,\dots,d\}$ and $j \in \mathbb{Z}$ or $j \in \mathcal{R}$ or $j \in \mathcal{B}_a$, we denote $v^{(a,j)} = v + L(a,j)$ the element obtained from v by adding $j.1_{\mathcal{B}_a}$ to the a's coordinate.*
3. *For $u_a \in \mathcal{B}_a$, we denote by $u_a 1_{\mathcal{P}}$ the product of the unit element $1_{\mathcal{P}}$ of $\mathcal{P}$ with the element of $\mathcal{P}$, identified with u_a, all coordinates of which are the unit element, except for the a's coordinate which is equal to u_a. If no ambiguity can occur, we shall omit the unit $1_{\mathcal{P}}$ and simply denote by u_a this element of $\mathcal{P}$.*
4. *We denote $|I| = \sum_{i=1,\dots,i} |I(a)|$ (with $|I(a)| = I(a)$ if $I(a) \geq 0$ and $|I(a)| = -I(a)$ if $I(a) < 0$), which is called the* order of I.
5. *Given $\alpha \in \mathbb{Z}^d$, we denote $I^{\alpha} = \prod_{a=1}^{d} ((I(a))^{\alpha_a} 1_{\mathcal{P}})$, which is called the* α's power of I *(possibly in a sub-ring of the product of the fields of fractions over the ring $\mathcal{B}_a$).*
6. *Given $\alpha \in \mathbb{R}^d$, we denote by $I^{[\alpha]}$, which is called the* coordinate by coordinate α's power of I *the vector, the coordinates of which (possibly in a sub-ring of the product of the fields of fractions over the ring $\mathcal{B}_a$) are given by $I^{[\alpha]} = \prod_{a=1}^{d} ((I(a))^{\alpha_a})$.*
7. *we denote $I! = \prod_{a=1}^{d} (I(a)!)$, where $I(a)! = \prod_{i \in \mathbb{N}, i.1_{\mathcal{A}_a} \leq I(a)} (i.1_{\mathcal{P}})$ The element $I! \in \mathcal{P}$ is called the* factorial of I.
8. *If $(J(a))_{a=1,\dots,d}$ is another multi-index, we denote by $((IJ)(a))_{a=1,\dots,d}$ the multi-sequence with $(IJ)(a) = I(a)J(a)$, which is called the* product of I and J.
9. *If, for $a = 1,\dots,d$, the algebra $\mathcal{B}_a$ is provided with an analyzable space structure and $\preceq_a$ is the order underlying this analyzable space structure, and if $(J(a))_{a=1,\dots,d}$ is another multi-index, we denote by $\leq$ the binary relation, which is a partial order, such that $I \leq J$ if and only if for $a = 1\dots,d$ we have $I(a) \preceq_a J(a)$.*
10. *If $(J(a))_{a=1,\dots,d}$ is another multi-index, we denote by $\prec$ the binary relation such that $I \prec J$ if and only if for $I \leq J$ and $I \neq J$.*
11. *If $(J(a))_{a=1,\dots,d}$ is another multi-index, we denote by $<$ the binary relation such that $I < J$ if and only if for for $a = 1\dots,d$ we have $I(a) < J(a)$.*
12. *We denote by 0 the multi-sequence with d coordinates equal to 0, and by 1 the multi-sequence with d coordinates equal to d. Note that the dimension d of these vectors can be omitted as, due to the context, no ambiguity will arise in practice.*

13. *If* $(J(a))_{a=1,\dots,d}$ *is another multi-index with* $J \le I$*, we denote by* $\binom{I}{J}$ *the element of the ring* $\mathcal{R}$ *defined by:*

$$\binom{I}{J} = \prod_{a=1}^{d} \binom{I(a)}{J(a)}$$

where $\binom{I(a)}{J(a)}$ *is the binomial coefficient defined as usual using the Pascal induction formula:*

$$\binom{I(a)}{J(a)} = 1 \text{ if } (I(a) = J(a) \text{ or } J(a) = 0_{\mathcal{B}_a}),$$

$$\binom{I(a)}{J(a)} = 0 \text{ if } (I(a) < J(a) \text{ or } J(a) < 0_{\mathcal{B}_a}),$$

$$\text{and } \binom{I(a)}{J(a)} = \binom{I(a) - 1\mathcal{B}_a}{J(a) - 1\mathcal{B}_a} + \binom{I(a) - 1\mathcal{B}_a}{J(a)} \text{ otherwise}$$

The element $\binom{I}{J}$ *is called the* multi-dimensional binomial coefficient of J from I.

Remark 2.4 (Multidimensional Pascal Formula) *Using Notation 2.1, we get for* $a = 1, \dots, d$ *the following multi-dimensional version of the Pascal Formula:*

$$\binom{I}{J} = \binom{I^{(a,-1)}}{J^{(a,-1)}} + \binom{I^{(a,-1)}}{J}$$

2.2.1 Digital Differentiation Masks and their Tensor Products

We now introduce a notion of digital differentiations.

Definition 2.3 [Digital differentiation Mask] Let $\omega \in \mathbb{N}^d$. A *(d−dimensional) digital* ω*−differentiation mask* is a multi-sequence $\mathbf{u} = (u(I))_{I \in \mathcal{Z}_d} \in \mathcal{M'}^{\mathcal{Z}_d}$ with finite support, satisfying the following properties:

1. For all $k \in \mathbb{N}^d$ with $0 \le k_a \le \omega_a$ and $k \ne \omega$, we have:

$$\sum_{I \in \mathcal{Z}_d} \left(\prod_{a=1}^{d} (I(a))^{k_a} \right) u(I) = 0_{\mathcal{M'}} \tag{1}$$

2.

$$\sum_{I \in \mathcal{Z}_d} \left(\prod_{a=1}^{d} (I(a))^{\omega_a} \right) u(I) = \prod_{a=1}^{d} ((-1_{\mathcal{M'}})^{\omega_a} \omega_a!) \tag{2}$$

Remark 2.5 *Using Notation 2.1, we can rewrite Definition 2.3 above saying that* $\mathbf{u}$ *is a (d−dimensional) digital* ω*−differentiation mask if and only if we have:*

$$\sum_{I \in \mathcal{Z}_d} I^k u(I) = 0_{\mathcal{M'}} \text{ for } 0 \le k \prec \omega \tag{3}$$

and

$$\sum_{I \in \mathcal{Z}_d} I^\omega u(I) = (-1_{\mathcal{M'}})^\omega \omega! \tag{4}$$

Definition 2.4 [Extended Digital differentiation Mask] Let $\omega \in \mathbb{N}^d$. An *extended (d−dimensional) digital ω−differentiation mask* is a rapidly decreasing multi-sequence $\mathbf{u} = (u(I))_{I\in\mathcal{Z}_d} \in \mathcal{M'}^{\mathcal{Z}_d}$ with finite support, satisfying Formula (3) and Formula (4).

In the sequel, unless otherwise specified, we shall say an $\omega-$differentiation mask as a shorthand for an extended ($d-$dimensional) digital $\omega-$differentiation mask.

Definition 2.5 [Tensor Product of Masks or Functions] For $a \in \{1,\dots,d\}$, let $\mathcal{I}_a$ be a subalgebra of $\mathcal{M}_a$ which contains $1_{\mathcal{A}_a}\mathcal{A}_a$ (typically $\mathcal{I}_a = 1_{\mathcal{A}_a}\mathcal{A}_a$ or $\mathcal{I}_a = \mathcal{A}_a$), and let $\mathbf{u}_a = (u_a(I))_{I\in\mathcal{I}_a} \in \mathcal{M'}^{\mathcal{I}_a}$ be a sequence with a one-dimensional domain $\mathcal{I}_a$. We denote $\mathcal{I} = \prod_{a=1}^d \mathcal{I}_a$ and $\mathbf{u} = (u(I))_{I\in\mathcal{I}} \in \mathcal{M'}^{\mathcal{I}}$ the multi-sequence defined by

$$u(I) = \prod_{a=1}^{d} u_a(I(a))$$

The function $\mathbf{u}$ is called the *tensor product of the function* $\mathbf{u}_a$ *for* $a = 1,\dots,d$. We denote by $\bigotimes_{a=1}^{d} \mathbf{u}_a \in \mathcal{M'}^{\mathcal{I}}$ the tensor product $\mathbf{u}$.

Definition 2.6 [Isotropic Multi-Sequence] Let us consider a multi-sequence $\mathbf{u} = (u(I))_{I\in\mathcal{Z}_d} \in \mathcal{M'}^{\mathcal{Z}_d}$. Let $a_1 \in \{1\dots,d\}$. For $I \in \mathcal{Z}_d$ and $i_1 \in \mathcal{A}_{a_1}$, we consider

$$(\tau_{a_1}(I,i_1))\,(a) = \begin{cases} I(a) \text{ if } a \neq a_1 \\ i_1 \text{ if } a = a_1 \end{cases}$$

thus defining an element $\tau_{a_1}(I,i_1)$ in $\mathcal{M'}^{\mathcal{Z}_d}$. The multi-sequence $\mathbf{u}$ is called *isotropic* if and only if for any $I \in \mathcal{Z}_d$, any $a_1 \in \{1\dots,d\}$, any $i_1 \in \mathcal{A}_{a_1}$, we have:

$$u\left(\tau_{a_1}(I,J(a_1))\right) u\left(\tau_{a_1}(J,I(a_1))\right) = u\left(I\right) u\left(J\right)$$

Proposition 2.2 *Let $\mathbf{u} = (u(I))_{I\in\mathcal{Z}_d} \in \mathcal{M'}^{\mathcal{Z}_d}$ be a multi-sequence. If $\mathbf{u}$ is a tensor product of one-dimensional sequences, that is, there exist $\mathbf{u}_a = (u_a(I))_{I\in\mathcal{A}_a} \in \mathcal{M'}^{\mathcal{A}_a}$ such that $\mathbf{u} = \bigotimes_{a=1}^{d} \mathbf{u}_a$. Then, it is isotropic.*

Proof. Assume that $\mathbf{u} = \bigotimes_{a=1}^{d} \mathbf{u}_a$ and consider $I \in \mathcal{Z}_d$ and $a_1 \in \{1\dots,d\}$.

$$\begin{aligned}
&u\left(\tau_{a_1}(I,J(a_1))\right) u\left(\tau_{a_1}(J,I(a_1))\right) \\
&= \left(\prod_{a=1}^d u_a\left(\tau_{a_1}(I,J(a_1))(a)\right)\right)\left(\prod_{a=1}^d u_a\left(\tau_{a_1}(J,I(a_1))(a)\right)\right) \\
&= u_{a_1}(J(a_1))\left(\prod_{a\neq a_1} u_a(I(a))\right) u_{a_1}(I(a_1))\left(\prod_{a\neq a_1} u_a(J(a)))\right) \\
&= \left(\prod_{a=1}^d u_a(I(a))\right)\left(\prod_{a=1}^d u_a(J(a))\right) \\
&= u(I)u(J)
\end{aligned}$$

□

Theorem 2.1 *A multi-sequence* $\mathbf{u} = (u(I))_{I\in\mathcal{Z}_d} \in \mathcal{M}'^{\mathcal{Z}_d}$ *is an isotropic digital* $\omega-$*differentiation mask if and only if, for* $a = 1,\dots,d$, *there exist one-dimensional* ω_a-*differentiation masks* $\mathbf{u}_a = (u_a(I))_{I\in\mathcal{Z}_1} \in \mathcal{M}'^{\mathcal{Z}_1}$ *such that* $\mathbf{u} = \bigotimes_{a=1}^{d} \mathbf{u}_a$.

Proof. Let us first prove by induction that a tensor product of d one-dimensional ω_a-differentiation masks is an isotropic differentiation mask. We already know from Proposition 2.2 that a tensor product of d one-dimensional masks is isotropic. We prove the result by induction on d. For $d = 1$, there is nothing to prove. Let us assume the result true for $d-1$ one-dimensional ω_a-differentiation masks, and consider, for $a = 1,\dots,d$, a one-dimensional ω_a-differentiation mask $\mathbf{u}_a = (u_a(I))_{I\in\mathcal{Z}_1} \in \mathcal{M}'^{\mathcal{Z}_1}$. Let $k \in \mathbb{N}^d$ with $0 \le k_a \le \omega_a$ and $k \neq \omega$. We have:

$$\begin{array}{rcl}
\sum_{I\in\mathcal{Z}_d}\left(\prod_{a=1}^d (I(a))^{k_a}\right)u(I) & = & \sum_{I\in\mathcal{Z}_d}\left(\prod_{a=1}^d (I(a))^{k_a}\right)\left(\prod_{a=1}^d u_a(I(a))\right)\\
& = & \sum_{i\in\mathcal{A}_d}\sum_{I\in\mathcal{Z}_{d-1}}\left(i^{k_d}\prod_{a=1}^{d-1}(I(a))^{k_a}\right)\left(u_d(i)\prod_{a=1}^{d-1}u_a(I(a))\right)\\
& = & \left(\sum_{i\in\mathcal{A}_d}(i^{k_d}u_d(i))\right)\left(\sum_{I\in\mathcal{Z}_{d-1}}\left(\prod_{a=1}^{d-1}(I(a))^{k_a}\left(\bigotimes_{a=1}^{d-1}\mathbf{u}_a\right)(I)\right)\right)\\
& = & 0_{\mathcal{M}'}
\end{array}$$

The last equality follows from our induction hypothesis, either applied to the one-dimensional ω_d-differentiation mask $\mathbf{u}_d$, either to the $(d-1)-$dimensional differentiation mask $\bigotimes_{a=1}^{d-1}\mathbf{u}_a$, depending on which of the coordinates of k differs form the corresponding coordinate of ω. Similarly,

$$\begin{array}{rcl}
\sum_{I\in\mathcal{Z}_d}\left(\prod_{a=1}^d (I(a))^{\omega_a}\right)u(I) & = & \sum_{I\in\mathcal{Z}_d}\left(\prod_{a=1}^d (I(a))^{\omega_a}\right)\left(\prod_{a=1}^d u_a(I(a))\right)\\
& = & \left(\sum_{i\in\mathcal{A}_d}(i^{\omega_d}u_d(i))\right)\left(\sum_{I\in\mathcal{Z}_{d-1}}\left(\prod_{a=1}^{d-1}(I(a))^{\omega_a}\left(\bigotimes_{a=1}^{d-1}\mathbf{u}_a\right)(I)\right)\right)\\
& = & \prod_{a=1}^{d-1}\left((-1_{\mathcal{M}'})^{\omega_a}\omega_a!\right)\left((-1_{\mathcal{M}'})^{\omega_d}\omega_d!\right) = \prod_{a=1}^{d-1}\left((-1_{\mathcal{M}'})^{\omega_a}\omega_a!\right)
\end{array}$$

Conversely, let us consider an isotropic digital $\omega-$differentiation mask $\mathbf{u} = (u(I))_{I\in\mathcal{Z}_d}$. Again, we prove the result by induction. If $d = 1$ there is nothing to prove. Assume the result true for a $(d-1)-$dimensional isotropic mask. For $I \in \mathcal{Z}_{d-1}$ and $i \in \mathcal{A}_d$, we set $u(I,i)$ the value of the $d-$dimensional multi-sequence $\mathbf{u}$ evaluated on $(I(1),\dots,I(d-1),i)$. For $i \in \mathcal{A}_d$, we set:

$$u_d(i) = \sum_{I\in\mathcal{Z}_{d-1}}\left(\frac{\prod_{a=1}^{d-1}(I(a))^{\omega_a}}{\prod_{a=1}^{d-1}(-1)^{\omega_a}\omega_a!}u(I,i)\right)$$

and, for $I \in \mathcal{Z}_{d-1}$,

$$u^{(d-1)}(I) = \sum_{i\in\mathcal{A}_d}\left(\frac{i^{\omega_d}u(I,i)}{(-1)^{\omega_d}\omega_d!}\right)$$

Both multi-sequences $\mathbf{u}_d$ and $\mathbf{u}^{(d-1)}$ are clearly isotropic. We show that $\mathbf{u} = \mathbf{u}^{(d-1)} \otimes \mathbf{u}_d$, that is: $u^{(d-1)}(J)u_d(j) = u(J,j)$. Indeed,

$$\begin{array}{rcl}
u^{(d-1)}(J)u_d(j) & = & \left(\sum_{i\in\mathcal{A}_d}\left(\frac{i^{\omega_d}u(J,i)}{(-1)^{\omega_d}\omega_d!}\right)\right)\left(\sum_{I\in\mathcal{Z}_{d-1}}\left(\frac{\prod_{a=1}^{d-1}(I(a))^{\omega_a}}{\prod_{a=1}^{d-1}(-1)^{\omega_a}\omega_a!}u(I,j)\right)\right)\\
& = & \sum_{i\in\mathcal{A}_d}\sum_{I\in\mathcal{Z}_{d-1}}\left(\frac{\prod_{a=1}^{d}(I(a))^{\omega_a}}{\prod_{a=1}^{d}(-1)^{\omega_a}\omega_a!}u(J,i)u(I,j)\right)\\
& = & \sum_{i\in\mathcal{A}_d}\sum_{I\in\mathcal{Z}_{d-1}}\left(\frac{\prod_{a=1}^{d}(I(a))^{\omega_a}}{\prod_{a=1}^{d}(-1)^{\omega_a}\omega_a!}\left(u(J,j)u(I,i)\right)\right)\\
& = & u(J,j)\sum_{i\in\mathcal{A}_d}\sum_{I\in\mathcal{Z}_{d-1}}\left(\frac{\prod_{a=1}^{d}(I(a))^{\omega_a}}{\prod_{a=1}^{d}(-1)^{\omega_a}\omega_a!}u(I,i)\right)\\
& = & u(J,j)
\end{array}$$

Let us now prove that, $\mathbf{u}_d$ is a one-dimensional differentiation mask. Let us set $k_a = \omega_a$ for $a = 1, \ldots, d-1$. Let $0 \leq k_d < \omega_d$, we have:

$$\begin{array}{rcl}\sum_{i\in\mathcal{A}_d} i^{k_d} u_d(i) & = & \sum_{i\in\mathcal{A}_d} i^{k_d}\left(\sum_{I\in\mathcal{Z}_{d-1}}\left(\frac{\prod_{a=1}^{d-1}(I(a))^{\omega_a}}{\prod_{a=1}^{d-1}(-1)^{\omega_a}\omega_a!}\right)u(I,i)\right)\\ & = & \frac{1}{\prod_{a=1}^{d-1}(-1)^{\omega_a}\omega_a!}\sum_{I\in\mathcal{Z}_d}\left(\prod_{a=1}^{d}(I(a))^{k_a}\right)u(I)\\ & = & 0_{M'}\end{array}$$

Similarly,

$$\begin{array}{rcl}\sum_{i\in\mathcal{A}_d} i^{\omega_d} u_d(i) & = & \sum_{i\in\mathcal{A}_d} i^{\omega_d}\left(\sum_{I\in\mathcal{Z}_{d-1}}\left(\frac{\prod_{a=1}^{d-1}(I(a))^{\omega_a}}{\prod_{a=1}^{d-1}(-1)^{\omega_a}\omega_a!}\right)u(I,i)\right)\\ & = & \frac{1}{\prod_{a=1}^{d-1}(-1)^{\omega_a}\omega_a!}\sum_{I\in\mathcal{Z}_d}\left(\prod_{a=1}^{d}(I(a))^{\omega_a}\right)u(I)\\ & = & (-1)^{\omega_d}\omega_d!\end{array}$$

Now we prove that that, if we set $\omega^{(d-1)} = (\omega_1, \ldots, \omega_{d-1})$, $\mathbf{u}^{(d-1)}$ is a $(d-1)-$dimensionnal $\omega^{(d-1)}-$differentiation mask. The result then follows from our induction hypothesis. Let $k \in \mathbb{N}^{d-1}$ with $0 \leq k_a \leq \omega_a^{(d-1)}$ for $a = 1, \ldots, d-1$ and $k \neq \omega^{(d-1)}$. We have

$$\begin{array}{rcl}\sum_{I\in\mathcal{Z}_{d-1}}\left(\prod_{a=1}^{d-1}(I(a))^{k_a}\right)u^{(d-1)}(I) & = & \sum_{I\in\mathcal{Z}_{d-1}}\left(\prod_{a=1}^{d-1}(I(a))^{k_a}\right)\sum_{i\in\mathcal{A}_d}\left(\frac{i^{\omega_d}u(I,i)}{(-1)^{\omega_d}\omega_d!}\right)\\ & = & \sum_{I\in\mathcal{Z}_d}\left(\prod_{a=1}^{d}(I(a))^{k_a}\right)u(I)\\ & = & 0_{\mathcal{M}'}\end{array}$$

Similarly,

$$\begin{array}{rcl}\sum_{I\in\mathcal{Z}_{d-1}}\left(\prod_{a=1}^{d-1}(I(a))^{\omega_a}\right)u^{(d-1)}(I) & = & \sum_{I\in\mathcal{Z}_{d-1}}\left(\prod_{a=1}^{d-1}(I(a))^{\omega_a}\right)\sum_{i\in\mathcal{A}_d}\left(\frac{i^{\omega_d}u(I,i)}{(-1)^{\omega_d}\omega_d!}\right)\\ & = & \frac{1}{(-1)^{\omega_d}\omega_d!}\sum_{I\in\mathcal{Z}_d}\left(\prod_{a=1}^{d}(I(a))^{\omega_a}\right)\\ & = & \prod_{a=1}^{d-1}\left((-1_{\mathcal{M}'})^{\omega_a}\omega_a!\right)\end{array}$$

□

In the sequel of this paper, all the considered differentiation masks are assumed to be isotropic.

2.2.2 Convolution and Differentiation Operators

Definition 2.7 [Convolution Product] Let $\mathbf{u}$ be a multi-sequence in ${\mathcal{M}'}^{\mathcal{Z}_d}$. Let $\mathcal{I}$ be a sub-algebra of $\mathcal{M}$ which contains $\mathcal{Z}_d$ (typically, $\mathcal{I} = \mathcal{Z}_d$ or $\mathcal{I} = \mathcal{M}$) and $\mathbf{v} : \mathcal{I} \longrightarrow \mathcal{M}'$ be a function. We say that $\mathbf{u}$ and $\mathbf{v}$ are *convolvable* if the following sum is absolutely convergent for any $N \in \mathcal{I}$:

$$(\mathbf{u} \star \mathbf{v})(N) = \sum_{I\in\mathcal{Z}_d} u(I)v(N-I)$$

The multi-sequence $\mathbf{u} \star \mathbf{v}$ thus defined is then called the *convolution product of* $\mathbf{u}$ *and* $\mathbf{v}$.

Proposition 2.3 *For $i = 1\ldots, m$, let $\mathbf{u}_i$ and $\mathbf{v}_i$ be two multi-sequences on a network $\mathcal{Z}^{(m)}$ in a Cartesian product of analyzable spaces $\mathcal{M}^{(m)}$, with values in $\mathcal{M}'$. Then, we have*

$$\left(\bigotimes_{i=1}^{m}\mathbf{u}_i\right) \star \left(\bigotimes_{i=1}^{m}\mathbf{v}_i\right) = \bigotimes_{i=1}^{m}(\mathbf{u}_i \star \mathbf{v}_i)$$

Proof. We prove the result for $m = 2$ and, by associativity of the tensor product and of the convolution product, the result follows by an immediate induction.

$$\begin{array}{rcl} ((u_1 \otimes u_2) \star (v_1 \otimes v_2))(n_1, n_2) & = & \sum_{(i_1,i_2)\in\mathbb{Z}^{(m_1)}\times\mathbb{Z}^{(m_2)}} (u_1 \otimes u_2)(i_1, i_2)(v_1 \otimes v_2)((n_1, n_2) - (i_1, i_2)) \\ & = & \sum_{(i_1,i_2)\in\mathbb{Z}^{(m_1)}\times\mathbb{Z}^{(m_2)}} (u_1(i_1)u_2(i_2)v_1(n_1 - i_1)v_2(n_2 - i_2)) \\ & = & \sum_{(i_1,i_2)\in\mathbb{Z}^{(m_1)}\times\mathbb{Z}^{(m_2)}} (u_1(i_1)v_1(n_1 - i_1))(u_2(i_2)v_2(n_2 - i_2)) \\ & = & \sum_{i_1\in\mathbb{Z}^{(m_1)}} (u_1(i_1)v_1(n_1 - i_1)) \sum_{i_2\in\mathbb{Z}^{(m_2)}} (u_2(i_2)v_2(n_2 - i_2)) \\ & = & ((u_1 \star v_1) \otimes (u_2 \star v_2))(n_1, n_2) \end{array}$$

□

Proposition 2.4 *For $i = 1\ldots, m$, let $\mathbf{u}_i$ be a multi-sequence on a network $\mathcal{Z}^{(i)}$ in a Cartesian product of analyzable spaces $\mathcal{M}^{(i)}$, with values in $\mathcal{M}'$. Then,*

1. *Suppose that for $i = 1, \ldots, m$, the multi-sequence is an $\omega^{(i)}-$differentiation mask. Then, the tensor product $\bigotimes_{i=1}^m \mathbf{u}_i$ is an $\omega-$differentiation mask, where ω is the concatenation of the vectors ω_i for $i = 1\ldots, m$.*

2. *Conversely, if we assume that $\mathbf{u} = \bigotimes_{i=1}^m \mathbf{u}_i$ is an $\omega-$differentiation mask on $\mathbb{Z} = \prod_{i=1}^m \mathcal{Z}^{(i)}$, where ω is the concatenation of the vectors ω_i for $i = 1\ldots, m$. then $\mathbf{u}_i$ is an $\omega^{(i)}-$differentiation mask for each $i \in \{1, \ldots, m\}$.*

Proof. **1)** We prove the first part of the result for $m = 2$ and, by associativity of the tensor product and vector concatenation, the result follows by an immediate induction. Let $\omega^{(1)} = (\omega_1^{(1)}, \ldots, \omega_{d_1}^{(1)})$ and $\omega^{(2)} = (\omega_1^{(2)}, \ldots, \omega_{d_2}^{(2)})$. Let $k_1 \in \mathbb{N}^{d_1}$ and $k_2 \in \mathbb{N}^{d_2}$ Let k be the concatenation of k_1 and k_2.

$$\begin{array}{l} \sum_{I\in\mathcal{Z}_d} \left(\prod_{a=1}^{d_1+d_2} (I(a))^{k_a}\right) (u_1 \otimes u_2)(I) \\ = \sum_{I\in\mathcal{Z}_{d_1}\times\mathcal{Z}_{d_2}} \left(\prod_{a=1}^{d_1+d_2} (I(a))^{k_a}\right) u_1(I_1)u_2(I_2)(I) \\ = \left(\sum_{I_1\in\mathcal{Z}_{d_1}} \left(\prod_{a=1}^{d_1} (I_1(a))^{k_a}\right) u_1(I_1)\right) \left(\sum_{I_2\in\mathcal{Z}_{d_2}} \left(\prod_{a=1}^{d_2} (I_2(a))^{k_a}\right) u_2(I_2)(I)\right) \end{array}$$

Then, depending on whether $k = \omega$ or not, we get Equation (1) or Equation (2).

2) To prove the converse, observe that $\mathbf{u}$ is not identically zero. Let $I \in \mathbb{Z}$ be such that $u(I) \neq 0_{\mathcal{M}'}$ and let $i \in \{1, \ldots, m\}$. By restricting $\mathbf{u}(I)$ to the elements $I \in \mathbb{Z}$ of the product $\mathbb{Z}$ in which only the i^{th} coordinate varies, we obtain a multi-sequence on $\mathcal{Z}^{(i)}$ which is proportional to $\mathbf{u}_i$. Then, Definition 2.4 applied to this restriction of $\mathbf{u}$ immediately yields Equation 1 and Equation 2 for $\mathbf{u}_i$. □

Definition 2.8 [Differentiation Operator] Let $\mathbf{u}$ be a differentiation mask with finite support. Let $\mathcal{I}$ be a sub-algebra of $\mathcal{M}$ with contains $\mathcal{Z}_d$. The *ω-differentiation operator associated to $\mathbf{u}$ over $\mathcal{M}^{\mathcal{I}}$* is the function $\Delta_{\mathbf{u}}$ with domain $\mathcal{M}'^{\mathcal{I}}$ and co-domain $\mathcal{R}^{\mathcal{I}}$ defined by

$$\Delta_{\mathbf{u}} : \left\{ \begin{array}{ccc} \mathcal{M}'^{\mathcal{I}} & \longrightarrow & \mathcal{M}'^{\mathcal{I}} \\ \mathbf{v} & \longmapsto & \Delta_{\mathbf{u}}(\mathbf{v}) = \mathbf{u} \star \mathbf{v} \end{array} \right.$$

Definition 2.9 [Extended Differentiation Operator] Let $\mathbf{u}$ be a rapidly decreasing differentiation mask. Let $\mathcal{I}$ be a sub-algebra of $\mathcal{M}$ with contains $\mathcal{Z}_d$. The *(extended) ω-differentiation operator associated to $\mathbf{u}$ over the space of moderately increasing functions $\mathcal{P}[\mathcal{I}, \mathcal{M}']$*, with codomain $\mathcal{P}[\mathcal{I}, \mathcal{M}']$, is defined by

$$\Delta_{\mathbf{u}} : \left\{ \begin{array}{ccc} \mathcal{P}[\mathcal{I}, \mathcal{M}'] & \longrightarrow & \mathcal{P}[\mathcal{I}, \mathcal{M}'] \\ \mathbf{v} & \longmapsto & \Delta_{\mathbf{u}}(\mathbf{v}) = \mathbf{u} \star \mathbf{v} \end{array} \right.$$

Remark 2.6 *Note that the fact that the image $\Delta_{\mathbf{u}}(\mathbf{v})$ with a rapidly decreasing sequence $\mathbf{u}$ and a moderately increasing function $\mathbf{v}$ lies in $\mathcal{P}[\mathcal{I}, \mathcal{M}']$ requires a justification, which is given in Proposition 2.7 shown below.*

In the sequel, if no ambiguity can arise, we shall assume without mentioning this hypothesis, either that differentiation masks have finite support, or the differentiation masks are rapidly decreasing and the corresponding differentiation operators are applied only to moderately increasing functions.

Proposition 2.5 *Let $\mathbf{u} = (u(I))_{i\in\mathcal{Z}_d}$ be an ω-derivative mask and $\mathbf{v} = (v(I))_{i\in\mathcal{Z}_d}$ be an ω'-derivative mask. Then $\mathbf{u} \star \mathbf{v}$ is an $\omega + \omega'$-derivative mask.*

Proof. We prove the one-dimensional case. The general case follows from Theorem 2.1 and Proposition 2.4.

Let $0 \le k \le \omega$. Then

$$\begin{array}{rcl} \displaystyle\sum_{n\in\mathbb{Z}} n^k(\mathbf{u}\star\mathbf{v})(n) & = & \sum_{n\in\mathbb{Z}}(i+(n-i))^k \sum_{i\in\mathbb{Z}} u(i)v(n-i) \\ & = & \sum_{n\in\mathbb{Z}}\sum_{i\in\mathbb{Z}}\sum_{p=0}^{k}\binom{k}{p} i^p(n-i)^{k-p}u(i)v(n-i) \\ & = & \sum_{p=0}^{k}\binom{k}{p}\left(\sum_{n\in\mathbb{Z}}\sum_{i\in\mathbb{Z}} i^p u(i)(n-i)^{k-p}v(n-i)\right) \\ & = & \sum_{p=0}^{k}\binom{k}{p}\sum_{j\in\mathbb{Z}}\sum_{i\in\mathbb{Z}} i^p u(i) j^{k-p}v(j) \\ & = & \sum_{p=0}^{k}\binom{k}{p}\left(\sum_{j\in\mathbb{Z}} j^{k-p}v(j)\right)\left(\sum_{i\in\mathbb{Z}} i^p u(i)\right) \end{array}$$

This is zero except if $k = \omega + \omega'$ and in this case all the terms are zero except if $p = \omega$ and in this case the sum is $\binom{\omega+\omega'}{\omega'}(-1)^{\omega}\omega!(-1)^{\omega'}\omega'! = (-1)^{\omega+\omega'}(\omega+\omega')!$ □

2.3 Differential Operators and Polynomials

Definition 2.10 [Canonical Morphisms X_a from $\mathcal{A}_a$ to $\mathcal{M}'$] For $a \in \{1, \ldots, d\}$, we consider X_a the unique morphism of algebra from $\mathcal{A}_a$ to $\mathcal{M}'$ such that the image of the unit element $1_{\mathcal{A}_a}$ in $\mathcal{A}_a$ is the unit element $1_{\mathcal{M}'}$ in $\mathcal{M}'$. The map X_a is called the *Canonical Morphisms X_a from $\mathcal{A}_a$ to $\mathcal{M}'$.*

The maps $\mathbf{p}_k$, for $k \in \mathbb{N}^d$ with $\sum_{a=1}^{d} k_a \le \delta$, defined by:

$$p_k(X_1, \ldots, X_d) = \prod_{a=1}^{d} X_a^{k_a} \tag{5}$$

are called *monomials* from $\mathcal{A}_a$ to $\mathcal{M}'$.

Definition 2.11 The $\mathcal{M}'-$*valued polynomial functions* with degree $\delta \in \mathbb{N}$ over $\mathcal{M}$ are the linear combinations of the monomials $\mathbf{p}_k$ introduced in Definition 2.10.

In other words, using Notation 2.1, any $\mathcal{M}'-$valued polynomial $\mathbf{p}$ with degree δ function over $\mathcal{M}$ is of the form

$$p(X) = \sum_{k\in\mathbb{N}^d, |k|\leq\delta} \lambda_k X^k \text{ where } X = (X_a)_{a=1,\ldots,d} \tag{6}$$

where $\lambda_k \in \mathcal{M}'$. The λ_k's, for $|k| \leq \delta$ are called the *coefficients of the polynomial* $\mathbf{p}$ *for the basis of the* $\mathbf{p}_k$.

Remark 2.7 *Let* $\pi_a(\mathbf{p}_k)$ *be the* $\mathcal{M}'-$*valued polynomial function over* $\mathcal{M}$ *defined by*

$$\pi_a(\mathbf{p}_k)(X_1, \ldots, X_d) = X_a^{k_a}$$

Then we have $\mathbf{p}_k = \bigotimes_{a=1}^{d} \pi_a(\mathbf{p}_k)$.

Proposition 2.6 *Let* $\mathbf{p}$ $\mathcal{M}'-$*valued polynomial function over* $\mathcal{M}$ *as defined in Equation (5). Let* $\mathbf{u} = \bigotimes_{a=1}^{m} \mathbf{u}_a$ *be an* $\omega-$*differentiation operator, with* $\omega \in \mathbb{N}^d$. *Then, we have*

$$(\Delta_{\mathbf{u}}(\mathbf{p})) = \sum_{i=0}^{\delta} \sum_{\substack{k\in\mathbb{N}^d \\ k_1+\ldots+k_d=i}} \left(\lambda_k \prod_{a=1}^{d} \binom{k_a}{k_a-\omega_a}\right) \prod_{a=1}^{d} X_a^{k_a-\omega_a} \tag{7}$$

In other words, differentiation operators act on polynomial functions like usual partial derivative operators on (say) usual polynomials over $\mathbb{R}^d$.

Remark 2.8 *Equation 7 can be rewritten using Notation 2.1 to obtain:*

$$(\Delta_{\mathbf{u}}(\mathbf{p}))(n) = \sum_{k\in\mathbb{N}^d, |k|\leq\delta} \lambda_k \binom{k}{k-\omega} X^{k-\omega} \tag{8}$$

Moreover, in the latter sum, only the multi-indices k *such that* $\omega \leq k$ *contribute with a non-zero term.*

Proof. Due to Proposition 2.3, remark 2.7 and Proposition 2.4, it is sufficient to prove the result for $d = 1$. By linearity, it is also sufficient to prove it for a monomial $\mathbf{p} = n^k$ Then,

$$\begin{aligned}(\Delta_{\mathbf{u}}(\mathbf{p}))(n) &= \textstyle\sum_{i\in\mathcal{A}_a} u(i)(n-i)^k \\ &= \textstyle\sum_{i\in\mathcal{A}_a} u(i) \sum_{l=0}^{k} \binom{k}{l} n^l (-i)^{k-l} \\ &= \textstyle\sum_{l=0}^{k} \binom{k}{l} \left(\sum_{i\in\mathcal{A}_a} u(i)(-i)^{k-l}\right) n^l\end{aligned}$$

Now, from Definition 2.4, the sum $\sum_{i\in\mathcal{A}_a} u(i)(-i)^{k-l}$ is equal to $0_{\mathcal{M}'}$ if $k-l < \omega_1$, and equal to $((-1)^{\omega_1}\omega_1!)$ if $k - l = \omega_1$. Hence, for $k \leq \omega_1$, $(\Delta_{\mathbf{u}}(\mathbf{p}))(n) = \binom{k}{k-\omega_1}((-1)^{\omega_1}\omega_1!)n^{k-\omega_1}$.

At last, we prove the result for any $k > \omega_1$ by induction. Suppose it is true for $k-1$, and set $\mathbf{v} = (v(n))_{n\in\mathcal{Z}_1}$, with $v(n) = \sum_{s\leq n} u(s)$. Then we have $\mathbf{u} = \Delta_- * \mathbf{v}$, where Δ_- is a finite difference $(1)-$differentiation mask (specifically: $\Delta_- * \mathbf{v}(n) = v(n) - v(n-1) = u(n)$). It can be seen that the mask $\mathbf{v}$ is a $k-1$ differentiation mask. Furthermore, the differential $\Delta_{\mathbf{u}}(\mathbf{p})$, which is a $(1)-$differential differentiation mask applied to the $(k-1)-$differential $\Delta_{\mathbf{v}}(\mathbf{p})$ which is constant (equal either to $((-1)^{\omega_1}\omega_1!)n^0$ if $k - 1 = \omega_1$ or, by induction hypothesis, identically zero otherwise), is also zero. $\square$

Lemma 2.4 *Let* $\mathbf{u}$ *be a rapidly decreasing function in* $\mathcal{M}'^{\mathcal{A}_a}$*, with* $a \in \{1,\ldots,d\}$*. Let* $\mathbf{p}$ *be a polynomial with degree* k *on a sub-algebra* $\mathcal{I}_a$ *of* $\mathcal{A}_a$ *containing* $1_{\mathcal{A}_a}\mathcal{A}_a$*. Then, there exists a polynomial function* π *over* $\mathcal{I}_a$ *with degree* k *such that:*

$$\mathbf{u} * \mathbf{p} = \pi$$

Proof. It is sufficient to prove the result for the monomial with degree k in $\mathbf{p}$. Hence we may assume w.l.o.g. that $p(i) = i^k$. We have: $\mathbf{u} * \mathbf{p}(n) = \sum_{i\in\mathcal{A}_a} u(i)(n-i)^k$. Now,

$$\begin{array}{rcl} \sum_{i\in\mathcal{A}_a} u(i)(n-i)^k & = & \sum_{i\in\mathcal{A}_a} u(i) \sum_{l=0}^{k} \binom{k}{l} n^l (-i)^{k-l} \\ & = & \sum_{l=0}^{k} \binom{k}{l} \left(\sum_{i\in\mathcal{A}_a} (-i)^{k-l} u(i)\right) n^l \end{array}$$

Due to Lemma 2.3, if we set $\pi(n) = \sum_{l=0}^{k} \binom{k}{l} \left(\sum_{i\in\mathcal{A}_a} (-i)^{k-l} u(i)\right) n^l$, the value $\pi(n)$ is well defined. The function π thus defined is a polynomial function of n, and we have $\mathbf{u} * \mathbf{p} \leq \pi$. □

Lemma 2.5 *Let* $\mathbf{u} = \bigotimes_{a=1}^{d} \mathbf{u}_i$ *be a rapidly decreasing function in* $\mathcal{M}'^{\mathcal{Z}_d}$*, with* $a \in \{1,\ldots,d\}$*. Let* $\mathbf{p}$ *be a polynomial with degree* $\delta \in \mathbb{N}$ *over a sub-algebra* $\mathcal{I}$ *of* $\mathcal{M}$ *containing* $\mathcal{Z}_d$*. Then, there exists a polynomial function* π *over* $\mathcal{I}$ *with degree* δ *over* $\mathcal{Z}_d$ *such that:*

$$\mathbf{u} * \mathbf{p} = \pi$$

Proof. Follows directly from Proposition 2.3, Remark 2.9, and Lemma 2.4. □

Remark 2.9 *Let* $\mathbf{u} = \bigotimes_{a=1}^{d} \mathbf{u}_a \in \mathcal{M}'^{\mathcal{Z}_d}$ *be a tensor product of non identically zero sequences. Then,* $\mathbf{u}$ *is rapidly decreasing if and only if* $\mathbf{u}_a$*'s is rapidly decreasing for each* $a \in \{1,\ldots,d\}$*.*

Proof. The "if part" is an immediate consequence of Lemma 2.5. The "only if" part is easily proved by distinguishing between the case when $\mathbf{u}$ is identically zero, in which case the result is obvious, and the case when $\mathbf{u}$ is not identically zero, in which case a restriction of $\mathbf{u}$ to a subset of $\mathcal{Z}_d$ where only one coordinate varies, which is proportional to $\mathbf{u}_a$, is seen to be rapidly decreasing. □

Similarly, we see:

Remark 2.10 *Let* $\mathbf{u} = \bigotimes_{a=1}^{d} \mathbf{u}_a \in \mathcal{M}'^{\mathcal{Z}_d}$ *be a tensor product of non identically zero sequences. Then,* $\mathbf{u}$ *is moderately increasing if and only if* $\mathbf{u}_a$*'s is moderately increasing for each* $a \in \{1,\ldots,d\}$*.*

Hence we have the following:

Proposition 2.7 *Let* $\mathbf{u} = \bigotimes_{a=1}^{d} \mathbf{u}_a$ *be a rapidly decreasing multi-sequence in* $\mathcal{M}'^{\mathcal{Z}_d}$*, with* $a \in \{1,\ldots,d\}$*. The convolution product* $\mathbf{u}$ *with a moderately increasing function over a sub-algebra* $\mathcal{I}$ *of* $\mathcal{M}$ *containing* $\mathcal{Z}_d$ *is always defined and is moderately increasing on* $\mathcal{I}$*.*

3 Multigrid Convergence for Differentials

The purpose of this section is to provide upper bounds for the difference between a digital derivative of a sampled (and quantized) signal, with possible errors on the values. We shall need a specific form of the Taylor Formula, in which we have an explicit form for the remainder, as in the integral form for the remainder. However, the formula we prove and use does not require that all partial derivatives of a given order be available or bounded. Instead, we assume that partial derivatives exist at different orders on the different variables, as, for example, in the tensor product of a C^2 function by a C^1 function, for which the differential of order $(2,1)$ exists and is continuous, but neither the differential of order $(1,2)$, nor the differential of order $(2,2)$ exist in general.

3.1 Taylor Formula with Multiple Integral Remainder

Definition 3.1 Let $x^{(1)} \leq x^{(2)}$ be two element of $\mathcal{M}$. We denote by $[x^{(1)}, x^{(2)}[$ the interval, set of all $T \in \mathcal{M}$ such that $x^{(1)} \leq T \leq x^{(2)}$. Let $X \in \{1, \ldots, d\}$ be a set of indices. We denote $\overline{X} = \{1, \ldots, d\} \backslash X$ the complement of X. We consider the following subsets of $\mathcal{M}$:

$$\mathcal{M}_X = \prod_{a \in X} \mathcal{A}_a \text{ and } \mathcal{M}_{\overline{X}} = \prod_{a \in \overline{X}} \mathcal{A}_a$$

$$\mathcal{C}_X(x^{(1)}, x^{(2)}) = \prod_{a \in X} [x_a^{(1)}, x_a^{(2)}[\text{ and } \mathcal{C}_{\overline{X}}(x^{(1)}, x^{(2)}) = \prod_{a \in \overline{X}} [x_a^{(1)}, x_a^{(2)}[$$

We have a clear identification through a natural isomorphism: $Id_X : \mathcal{M}_X \times \mathcal{M}_{\overline{X}} \longrightarrow \mathcal{M}$. We denote by T_X and [respectively $T_{\overline{X}}$] the projections of an element $T \in \mathcal{M}$ onto $\mathcal{M}_X$ [respectively $\mathcal{M}_{\overline{X}}$]. In that way, a function $f : [x^{(1)}, x^{(2)}[\longmapsto \mathcal{M}'$ can also be identified to a function

$$f_X : \left\{ \begin{array}{ccl} \mathcal{C}_X(x^{(1)}, x^{(2)}) \times \mathcal{C}_{\overline{X}}(x^{(1)}, x^{(2)}) & \longmapsto & \mathcal{M}' \\ (T, U) & \longrightarrow & f(Id_X(T, U)) \quad = f(T + U) \end{array} \right.$$

The sets $\mathcal{C}_X(x^{(1)}, x^{(2)})$ [respectively $\mathcal{C}_{\overline{X}}(x^{(1)}, x^{(2)})$] is called the *$X-$slice of the cube* $[x^{(1)}, x^{(2)}[$ [respectively the *$\overline{X}-$slice of the cube* $[x^{(1)}, x^{(2)}[$].

For each $a \in \{1, \ldots, d\}$, we consider dt_a the measure on $\mathcal{A}_a$ underlying the analyzable space structure. We consider $dT_X = \prod_{a \in X} dt_a$ the product measure on $\mathcal{M}_X$. At last, we consider the operator

$$Int_X : \left\{ \begin{array}{ccc} L_1([x_a^{(1)}, x_a^{(2)}], \mathcal{M}') & \longmapsto & L_1(C_{\overline{X}}, \mathcal{M}') \\ f & \longmapsto & Int_X(f) \end{array} \right.$$

with, for $f \in L_1([x^{(1)}, x^{(2)}], \mathcal{M}')$ and $U \in C_{\overline{X}}$,

$$\left(Int_X(f)\right)(U) = \int_{\mathcal{C}_X(x^{(1)}, x^{(2)})} f(T_X + U) dT_X$$

The function $Int_X(f)$ is called the *partial integral of f over the $X-$slices of the cube* $[x^{(1)}, x^{(2)}[$. By convention, if $X = \emptyset$, the integral $\int_{\mathcal{C}_X(x^{(1)}, x^{(2)})} f_X(T_X, U) dT_X$ is defined equal to $f(U)$, so that $Int_\emptyset(f) = f$.

Notation 3.1 *Let $X \subset \{1, \ldots, d\}$. We denote $\mathbb{1}_X \in \mathbb{N}^d$ the vector such that for $a = 1, \ldots, d$, the coordinate $(\mathbb{1}_X)_a$ is equal to 1 if $a \in X$, and is equal to 0 otherwise.*

Theorem 3.1 (Taylor Formula with Multiple Integral Remainder) *Let $f : \mathcal{M} \longrightarrow \mathcal{M}'$ be a map and let $\omega \in \mathbb{N}^d$. We assume that the partial differentials $f^{(J)}$ of the map f exist and are continuous for all $J \in \mathbb{N}^d$ with $0 \leq J \leq \omega + 1$. Then, using Notation 2.1, we have the following identity, for x and $x^{(0)}$ in $\mathcal{M}$:*

$$f(x) = \sum_{X \subset \{1,\dots,d\}} \sum_{J \in \{\omega_X\} \times [0, \omega_{\overline{X}}]} \int_{\mathcal{C}_X(x^{(0)},x)} f^{(J+\mathbb{1}_X)} \left(T_X + x^{(0)}_{\overline{X}}\right) \frac{\left(x - x^{(0)}_{\overline{X}} - T_X\right)^J}{J!} dT_X \qquad (9)$$

where, in accordance with Definition 3.1, the set $\{\omega_X\} \times [0, \omega_{\overline{X}}]$ denotes the set of all $J \in \mathbb{N}^d$ such that $0 \leq J \leq \omega$ and such that $J_a = \omega_a$ for all $a \in X$. This identity is called the generalized Taylor formula with integral form for the remainder.

Proof. We prove the result by induction on d. For $d = 1$, there are two possible subsets $X \subset \{1, \dots, d\}$: $X = \emptyset$ and $X = \{1\}$.

The term for $X = \emptyset$ yields

$$\sum_{J \in [0,(\omega_1)]} f^{(J+\mathbb{1}_\emptyset)} \left(x^{(0)}_{\{1\}}\right) \frac{\left(x - x^{(0)}_{\{1\}}\right)^J}{J!} = \sum_{j=0}^{\omega_1} f^{(j)}(x^{(0)}) \frac{(x - x^{(0)})^j}{j!}$$

The term for $X = \{1\}$ yields

$$\sum_{J \in \{\omega_1\}} \int_{\mathcal{C}_{\{1\}}(x^{(0)},x)} f^{(J+\mathbb{1}_{\{1\}})} \left(T_{\{1\}}\right) \frac{\left(x - T_{\{1\}}\right)^J}{J!} dT_{\{1\}} = \int_{x^{(0)}}^{x} f^{(\omega_1+1)}(T) \frac{(x - T)^{\omega_1}}{\omega_1!} dT$$

Hence Equation 9 corresponds for $d = 1$ to the usual Taylor Theorem with Integral Remainder in $1D$

$$f(x) = \sum_{j=0}^{\omega_1} f^{(j)}(x^{(0)}) \frac{(x - x^{(0)})^j}{j!} + \int_{x^{(0)}}^{x} f^{(\omega_1+1)}(T) \frac{(x - T)^{\omega_1}}{\omega_1!} dT$$

which is proved as usual.

Now, assume that the result is true in dimension $d - 1$, with $d \geq 2$. We consider the element $x_{\{1,\dots,d-1\}} + x^{(0)}_{\{d\}}$ of $\mathcal{M}$, all coordinates of which are equal to those of x, except the d^{th} coordinate which is equal to $x^{(0)}_d$

From the $1D$ case, dealt with above, applied to the value of $f(x) = f\left(x_{\{1,\dots,d-1\}} + x_{\{d\}}\right)$ expressed through the Taylor development of f at the point $x_{\{1,\dots,d-1\}} + x^{(0)}_{\{d\}}$, we get:

$$\begin{aligned} f\left(x_{\{1,\dots,d-1\}} + x_{\{d\}}\right) &= \sum_{j=0}^{\omega_d} f^{(j\mathbb{1}_{\{d\}})} \left(x_{\{1,\dots,d-1\}} + x^{(0)}_{\{d\}}\right) \frac{\left(x_{\{1,\dots,d-1\}} + x_{\{d\}} - x^{(0)}_{\{d\}}\right)^{j\mathbb{1}_{\{d\}}}}{j!} \\ &+ \int_{x^{(0)}_{\{d\}}}^{x_{\{d\}}} f^{((\omega_d+1)\mathbb{1}_{\{d\}})}(x_{\{1,\dots,d-1\}} + T_{\{d\}}) \frac{\left(x_{\{1,\dots,d-1\}} + x_{\{d\}} - T_{\{d\}}\right)^{\omega_d \mathbb{1}_{\{d\}}}}{\omega_d!} dT_{\{d\}} \end{aligned}$$

From our induction hypothesis applied, for $j = 1, \dots, \omega_d + 1$, to the function

$$g_j : \left\{ \begin{array}{c} \mathcal{M}_{\{1,\dots,d\}} \longrightarrow \mathcal{M}' \\ y \longmapsto f^{(j\mathbb{1}_{\{d\}})} \left(y + x^{(0)}_{\{d\}}\right) \end{array} \right.$$

we have:

$$f^{(j\mathbb{1}_{\{d\}})}\left(x_{\{1,\dots,d-1\}}+x^{(0)}_{\{d\}}\right) = \sum_{X\subset\{1,\dots,d-1\}}\ \sum_{J\in\{\omega_X\}\times[0,\omega_{\overline{X}}]}\int_{\mathcal{C}_X\left(x^{(0)}_{\{1,\dots,d-1\}},x_{\{1,\dots,d-1\}}\right)} f^{(J+\mathbb{1}_X+j\mathbb{1}_{\{d\}})}\left(T_X+x^{(0)}_{\overline{X}}+x^{(0)}_{\{d\}}\right)\frac{\left(x_{\{1,\dots,d-1\}}-x^{(0)}_{\overline{X}}-T_X\right)^J}{J!}dT_X$$

Note that, as opposed to our statement in Equation 9, in the latest formula, $\overline{X}$ denotes the complement of X in $\{1,\dots,d-1\}$, as it is an application of our induction hypothesis in dimension $d-1$. By substituting the last expression for $f^{(j\mathbb{1}_{\{d\}})}\left(x_{\{1,\dots,d-1\}}+x^{(0)}_{\{d\}}\right)$ (substitution which is also valid, by changing $x^{(0)}_{\{d\}}$ for $T_{\{d\}}$, for $f^{((\omega_d+1)\mathbb{1}_{\{d\}})}\left(x_{\{1,\dots,d-1\}}+T_{\{d\}}\right)$), into the expression of $f(x)=f\left(x_{\{1,\dots,d-1\}}+x_{\{d\}}\right)$ above, we obtain:

$$\begin{aligned}
f(x) &= \sum_{j=0}^{\omega_d}\ \sum_{X\subset\{1,\dots,d-1\}}\ \sum_{J\in\{\omega_X\}\times[0,\omega_{\overline{X}}]}\int_{\mathcal{C}_X\left(x^{(0)}_{\{1,\dots,d-1\}},x_{\{1,\dots,d-1\}}\right)} f^{(J+\mathbb{1}_X)}\left(T_X+x^{(0)}_{\overline{X}}+x^{(0)}_{\{d\}}\right)\\
&\quad \frac{\left(x_{\{1,\dots,d-1\}}-x^{(0)}_{\overline{X}}-T_X\right)^J}{J!}\frac{\left(x_{\{1,\dots,d-1\}}+x_{\{d\}}-x^{(0)}_{\{d\}}\right)^{j\mathbb{1}_{\{d\}}}}{j!}dT_X\\
&\quad + \sum_{X\subset\{1,\dots,d\},d\in X}\ \sum_{J\in\{\omega_X\}\times[0,\omega_{\overline{X}}]}\int_{\mathcal{C}_X\left(x^{(0)},x_{\{1,\dots,d-1\}}+x^{(0)}_{\{d\}}\right)}\int_{x^{(0)}_d}^{x_d}\\
&\quad f^{(J+\mathbb{1}_X+(\omega_d+1)\mathbb{1}_{\{d\}})}\left(T_{\{d\}}+T_X+x^{(0)}_{\overline{X}}+x^{(0)}_{\{d\}}\right)\frac{\left(x_{\{1,\dots,d-1\}}-x^{(0)}_{\overline{X}}-T_X\right)^J}{J!}\\
&\quad \frac{\left(x_{\{1,\dots,d-1\}}+x_{\{d\}}-T_{\{d\}}\right)^{j\mathbb{1}_{\{d\}}}}{j!}dT_{\{d\}}dT_X\\
&= \sum_{X\subset\{1,\dots,d\}}\ \sum_{J\in\{\omega_X\}\times[0,\omega_{\overline{X}}]}\int_{\mathcal{C}_X\left(x^{(0)},x\right)} f^{(J+\mathbb{1}_X)}\left(T_X+x^{(0)}_{\overline{X}}\right)\frac{\left(x-x^{(0)}_{\overline{X}}-T_X\right)^J}{J!}dT_X
\end{aligned}$$

□

In the latest expression, $\overline{X}$ now denotes the complement of X in $\{1,\dots,d\}$.

3.2 Digitization, Quantization, Noise Models

In the sequel of this section, we consider a map $f:\mathcal{M}\longmapsto\mathcal{M}'$, and a map $\Gamma:\mathcal{Z}_d\longmapsto\mathcal{M}'$ on the discrete network $\mathcal{Z}_d$. We shall make extensive use of Notation 2.1 for exponentiation notations, as well as orders in $\mathcal{M}$ and $\mathcal{M}'$.

Let $h=(h_a)_{a\in\{1,\dots,d\}}\in\mathcal{M}$, with $0_{\mathcal{A}_a}<h_a$, be a strictly positive vector representing some digitization step in the domain of f. For $K=(k_a)_{a=1,\dots,d}\in\mathbb{R}^d$, we consider the element $h^{[K]}\in\mathcal{M}$. As in the definition of monomials (Definition 2.10) let $(h')^{[K]}_a\in\mathcal{M}'$ the image of $h^{[K]}_a$ by the unique morphism of algebra sending $1_{\mathcal{A}_a}$ to $1_{\mathcal{M}'}$. Since $h_a>0_{\mathcal{A}_a}$, we also have $(h')^{[K]}_a>0_{\mathcal{M}'}$. At last, we denote $h'=(h')^{[1]}$, corresponding to the case when $k_a=1$ for all $a\in\{1,\dots,d\}$.

By abuse, we shall write h instead of h' in some formulas, having in mind that, when considered as an element of $\mathcal{M}'$, a monomial function has been applied to the element $h\in\mathcal{M}$.

Definition 3.2 We say that the map Γ is a *digitization of f with error* $\varepsilon_{h,h'} : \mathcal{Z}_d \longrightarrow \mathcal{M}'$ if for any $N \in \mathcal{Z}_d$, setting as usual $(N.h)(a) = N(a)h_a$, and considering the element $(h'\Gamma(N)) = \left(\prod_{a=1}^d h'_a\right)(\Gamma(N))$ of $\mathcal{M}'$, we have:

$$h'\Gamma(N) = f(N.h) + \varepsilon_{h,h'}(N) \tag{10}$$

Definition 3.3 [Vector Valued Infinite Norm for Functions] Let $X \subset \mathcal{M}$ and let $g : X \longrightarrow \mathcal{M}'$ be a bounded function. The *infinte norm of* g, denoted by $\|g\|_\infty$ the vector in $\mathcal{M}'$ is defined as follows. For $a' \in \{1, \dots, d'\}$, we denote $N_{a'} = \sup_{x\in X}(|(g(x))_{a'}|)$ the upper bound of the $(a')^{th}$ coordinate of $g(x)$ in $\mathcal{M}'$. Now, we set

$$\|g\|_\infty = (N_{a'})_{a'=1,\dots,d'}$$

We consider the following particular models for the errors $\varepsilon_{h,h'}$ on the values:

- *Exact Values:* In this model, the values are known exactly:

  $$\varepsilon_{h,h'} \equiv 0_{M'}$$

  Note that, although this model has been the most widely used in approximation theory, this value error model is not very realistic from an Information Sciences point of view.

- *Uniform Noise (or Uniform Bias) on Values:* In this model, the error $\varepsilon_{h,h'}$ on the values is uniformly bounded by some constant which depends on the quantization step h'. In our model, however, this bound can be asymptotically greater that h'. Namely we assume here that (see Notation 2.1 for the coordinates by coordinates exponentiation, denoted with brackets notation)

  $$0 \leq |\varepsilon_{h,h'}(I)| \leq K(h')^{[\alpha]}$$

  where $\alpha \in \mathbb{R}^d$ with $0 < \alpha_a \leq 1$ for all $a \in \{1, \dots, d\}$, and K is a positive constant. Note that this error can also have some bias, in the sense that the average noise value (or expected value) could be non-zero.

- *Quantization of Values:* In this model, the errors $\varepsilon_{h,h'}$ on the values is uniformly bounded by $\frac{1}{2}h'$. This is a particular case of uniform noise with $\alpha = 1$, and corresponds to the case when some basic quantization has been obtained by rounding-off the exactly known values of the function, for example for digital storage. This case is equivalent to $\Gamma(I) = \left[\frac{f(Ih)}{h'}\right]$. A variant is when quantization has been obtained by an integer part (floor case): $0 \leq \varepsilon_{h,h'}(I) < h'$, which is equivalent to $\Gamma(I) = \lfloor\frac{f(Ih)}{h'}\rfloor$.

- *Stochastic Noise on Values*: In this model, the errors $\varepsilon_{h,h'}(I)$ on the different values for $I \in \mathcal{Z}_d$ are independent random variables with expected value 0 and standard deviation $\sigma(h')$, converging to 0 along with h'. In that case, Equation 10 implies that the values $\Gamma(I)$, for $I \in \mathcal{Z}_d$ also are defined as independent random variables.

3.3 Basic Error Decomposition and Upper Bounds

3.3.1 Errors Related to Sampling and to Input Values

In order to show that the digital $\omega-$differentiation of a digitization Γ of a real function f provides an estimate for the continuous derivative $f^{(\omega)}$ of f, we would like to evaluate, at each sample point $N \in \mathcal{Z}_d$, the difference between the digital differentiation $\frac{1}{(h')^{[\omega-1]}}(\Delta_{\mathbf{u}}^{\omega} \star \Gamma)(N)$ (where, as usual in this context, the product [resp. exponentiation] between two $d-$dimensional vectors is a coordinate by coordinate product [resp. exponentiation]) of the digitized signal and the value of the usual ω^{th} partial derivative $f^{(\omega)}(Nh)$ of f. This difference may easily be decomposed from Equation (10) and Definition 2.4 into the sum

$$\frac{1}{(h')^{[\omega-1]}}(\Delta_{\mathbf{u}}^{\omega} \star \Gamma)(N) - f^{(\omega)}(Nh) = ES_{\omega}(f, h, h', \Gamma, \mathbf{u}, N) + EV_{\omega}(f, h, h', \Gamma, \mathbf{u}, N) \tag{11}$$

where

$$ES_{\omega}(f, h, h', \Gamma, \mathbf{u}, N) = \left(\frac{1}{(h')^{[\omega]}} \sum_{I \in \mathcal{Z}_d} u(I) f\left((N-I)h\right) \right) - f^{(\omega)}(Nh) \tag{12}$$

is called the *sampling error*, and

$$EV_{\omega}(f, h, h', \Gamma, \mathbf{u}, N) = \frac{1}{(h')^{[\omega]}} \sum_{I \in \mathcal{Z}_d} u(I) \varepsilon_{h,h'}(N-I) \tag{13}$$

is called the *(input) values error*. As their names imply, the sampling error is due to the fact that we only know about the values of f at some grid points, and the values error is due to the fact that we do not know the exact values of f at sample points.

The sampling error is a real values sequence. Under the uniform bias hypothesis, the values error is also a real valued sequence, but under the stochastic hypothesis, the values error is a sequence of random variable.

3.3.2 Upper Bound for the Sampling Error

In the following lemma, we show that the sampling error can be bounded independently from the error on input values, using the mask values, the norm of the partial derivatives of f with order higher than ω^{th}, and a the digitization step. The immediate consequences are some convergence results in the case when exact values of the function at sample points are known.

Lemma 3.1 *Let us assume that the partial derivative $f^{(K)}$ exists and is continuous on $\mathcal{M}$, for every $K = (k_a)_{a=1,\dots,d} \in \mathbb{N}^d$ with $k_a \geq 1 + \omega_a$ for $a = 1, \dots, d$. Let $\mathbf{u}$ be a digital $\omega-$differentiation mask with convergence order ρ. Let $S = (s_1, \dots, s_d)$ with $s_a = max\{\omega_a, 1 + \rho_a\}$ for $a = 1, \dots, d$. Let Γ be a digitization of f with error $\varepsilon_{h,h'} : \mathcal{Z}_d \longrightarrow \mathcal{M}'$. Suppose that $f^{(s)}$ is bounded on $\mathbb{R}$. Then for all $N \in \mathcal{Z}_d$,*

$$ES_{\omega}(f, h, h', \Gamma, \mathbf{u}, N) \leq \sum_{X \subset \{1,\dots,d\}, X \neq \emptyset} ||f^{(\omega + \mathbb{1}_X)}||_{\infty} \sum_{I \in (\mathcal{Z}_d)_X} |I^{\omega_X + \mathbb{1}_X} u(I_X)| \frac{h^{\mathbb{1}_X}}{\omega_X!} \tag{14}$$

Moreover, if we consider a lowest order approximation when all coordinates of h tend to zero at the same speed (e.g. constant ratio), the error can be approximated by the sum for X

with cardinality 1*, which yields:*

$$ES_\omega(f,h,h',\Gamma,\mathbf{u},N) = O\left(\sum_{a=1}^{d} h_a\,||f^{(\omega+\mathbb{1}_{\{a\}})}||_\infty \sum_{I\in\mathcal{A}_a} |I^{\omega_a+1}\,u(I_X)|\right) \tag{15}$$

Proof. From the Taylor formula with Integral Remainder (see Theorem 3.1, the sum involved in Equation (12) can be written

$$\begin{array}{rcl}\sum_{I\in\mathcal{Z}_d} u(I)f\left((N-I)h\right) & = & \sum_{I\in\mathcal{Z}_d} u(I)\left[\sum_{X\subset\{1,\dots,d\}}\sum_{J\in\{\omega_X\}\times[0,\omega_{\overline{X}}]}\int_{\mathcal{C}_X(Nh,(N-I)h)}\right.\\ & & \left. f^{(J+\mathbb{1}_X)}\left(T_X+(Nh)_{\overline{X}}\right)\frac{\left((N-I)h-(Nh)_{\overline{X}}-T_X)\right)^J}{J!}dT_X\right]\end{array}$$

Now, for $X\subset\{1,\dots,d\}$ and $J\in\{\omega_X\}\times[0,\omega_{\overline{X}}$, we have

$$\begin{array}{rcl}\frac{\left((N-I)h-(Nh)_{\overline{X}}-T_X)\right)^J}{J!} & = & \frac{(((N-I)h)_X-T_X))^{J_X}}{J_X!}\frac{\left(((N-I)h)_{\overline{X}}-(Nh)_{\overline{X}})\right)^{J_{\overline{X}}}}{J_{\overline{X}}!}\\ & = & \frac{(((N-I)h)_X-T_X))^{J_X}}{J_X!}\frac{\left(-I_{\overline{X}}\right)^{J_{\overline{X}}}h_{\overline{X}}^{J_{\overline{X}}}}{J_{\overline{X}}!}\end{array}$$

Hence

$$\begin{array}{rcl}\sum_{I\in\mathcal{Z}_d} u(I)f\left((N-I)h\right) & = & \sum_{X\subset\{1,\dots,d\}}\sum_{J\in\{\omega_X\}\times[0,\omega_{\overline{X}}]} h_{\overline{X}}^{J_{\overline{X}}}\left[\sum_{I\in\mathcal{Z}_d} u(I)\frac{\left(-I_{\overline{X}}\right)^{J_{\overline{X}}}}{J_{\overline{X}}!}\right.\\ & & \left.\int_{\mathcal{C}_X(Nh,(N-I)h)} f^{(J+\mathbb{1}_X)}\left(T_X+(Nh)_{\overline{X}}\right)\frac{(((N-I)h)_X-T_X)^{J_X}}{J_X!}dT_X\right]\end{array}$$

Yet, since u is a tensor product due to Theorem 2.1, for $X\subset\{1,\dots,d\}$, we have $u(I) = u(I_X)u(I_{\overline{X}})$, where $I_X(a) = I(a)$ if $a\in X$ and $I_X(a) = 1_{\mathcal{A}_a}$ otherwise (and similarly for $I_{\overline{X}}$). Furthermore, due to $I_X\mapsto u(I_X)$ is an ω_X-differentiation mask, and $I_{\overline{X}}\mapsto u(I_{\overline{X}})$ is an $\omega_{\overline{X}}-$differentiation mask. Therefore,

$$\begin{array}{rcl}\sum_{I\in\mathcal{Z}_d} u(I)f\left((N-I)h\right) & = & \sum_{X\subset\{1,\dots,d\}}\sum_{J\in\{\omega_X\}\times[0,\omega_{\overline{X}}]} h_{\overline{X}}^{J_{\overline{X}}}\left[\sum_{I\in\mathcal{Z}_d} u(I_X)u(I_{\overline{X}})\frac{\left(-I_{\overline{X}}\right)^{J_{\overline{X}}}}{J_{\overline{X}}!}\right.\\ & & \left.\int_{\mathcal{C}_X(Nh,(N-I)h)} f^{(J+\mathbb{1}_X)}\left(T_X+(Nh)_{\overline{X}}\right)\frac{(((N-I)h)_X-T_X)^{J_X}}{J_X!}dT_X\right]\\ & = & \sum_{X\subset\{1,\dots,d\}}\sum_{J\in\{\omega_X\}\times[0,\omega_{\overline{X}}]} h_{\overline{X}}^{J_{\overline{X}}}\left[\sum_{I\in(\mathcal{Z}_d)_{\overline{X}}} u(I_{\overline{X}})\frac{\left(-I_{\overline{X}}\right)^{J_{\overline{X}}}}{J_{\overline{X}}!}\right]\\ & & \left[\sum_{I\in(\mathcal{Z}_d)_X} u(I_X)\int_{\mathcal{C}_X(Nh,(N-I)h)} f^{(J+\mathbb{1}_X)}\left(T_X+(Nh)_{\overline{X}}\right)\right.\\ & & \left.\frac{(((N-I)h)_X-T_X)^{J_X}}{J_X!}dT_X\right]\\ & = & \sum_{X\subset\{1,\dots,d\}} h\overline{X}^{\omega_{\overline{X}}}\sum_{I\in(\mathcal{Z}_d)_X} u(I_X)\\ & & \int_{\mathcal{C}_X(Nh,(N-I)h)} f^{(\omega+\mathbb{1}_X)}\left(T_X+(Nh)_{\overline{X}}\right)\frac{(((N-I)h)_X-T_X)^{\omega_X}}{\omega_X!}dT_X\end{array}$$

The last equality comes from the fact that, due to the fact that $I_{\overline{X}}\mapsto u(I_{\overline{X}})$ is an $\omega_{\overline{X}}-$differentiation mask (hence satisfies Equation (3) and Equation (4), all terms of the sums over $I\in(\mathcal{Z}_d)_{\overline{X}}$ between brackets are zero except for the term with $J_{\overline{X}} = \omega_{\overline{X}}$, form which Equation (4) holds. Finally, Now, using the expression for the sampling error (Equation 12), the term of the latest sum corresponding to $X = \emptyset$ cancels out with $-f^{(\omega)}(Nh)$, and we provide an upper bound for the remaining sum for $X \neq \emptyset$:

$$ES_\omega(f, h, h', \Gamma, \mathbf{u}, N) \leq \sum_{X \subset \{1,\ldots,d\}, X \neq \emptyset} ||f^{(\omega+\mathbb{1}_X)}||_\infty \sum_{I \in (\mathcal{Z}_d)_X} |I^{\omega_X+\mathbb{1}_X} u(I_X)| \frac{h_X^{\omega_X+\mathbb{1}_X} h_{\overline{X}}^{(h')^\omega \omega_{\overline{X}}}}{\omega_X!}$$

from which the result follows by simplification by $(h')^\omega$. □

Remark 3.1 *Lemma 3.1 shows that the sampling error tends to zero along with h for a fixed function and a fixed differentiation mask.*

3.3.3 Upper Bound for the Input Values Error

The following lemma gives an upper bound for the error related to uniform noise or uniform bias on the values at sample points (see Section 3.2).

Lemma 3.2 *Let* $\mathbf{u}$ *be a digital* $\omega-$*differentiation mask with convergence order* ρ*. Let us assume that* Γ *is a digitization of* f *with errors on input values* $\varepsilon_{h,h'}$ *such that* $\|\varepsilon_{h,h'}\|_\infty \leq K(h')^{[\alpha]}$ *with* $0 < \alpha_a \leq 1$ *for* $a = 1, \ldots, d$*, which satisfies the uniform noise/bias error model. Then, for all* $N \in \mathcal{Z}_d$*,*

$$|EV_\omega(f, h, h', \Gamma, \mathbf{u}, N)| \leq \frac{K}{(h')^{[\omega-\alpha]}} \left(\sum_{I \in \mathcal{Z}_d} |u(I)| \right)$$

Proof. We derive an upper bound for the values error from its expression in Equation (13):

$$|EV_\omega(f, h, h', \Gamma, \mathbf{u}, N)| \leq \frac{\|\varepsilon_{h,h'}\|_\infty}{(h')^{[\omega]}} \sum_{I \in \mathcal{Z}_d} |u(I)| \leq \frac{K}{(h')^{[\omega-\alpha]}} \left(\sum_{I \in \mathcal{Z}_d} |u(I)| \right)$$

□

The following lemma gives an upper bound for the error related to statistic noise with expected values 0 on the values at sample points.

Lemma 3.3 *Let* $\mathbf{u}$ *be a digital* $\omega-$*differentiation mask. Assume that* Γ *is a digitization of* f *with error on input values* $\varepsilon_{h,h'}$ *following the stochastic noise model. In other words, the* $\varepsilon_{h,h'}(N)$*'s for all* $N \in \mathcal{Z}_d$ *are independent random variable with expected value* 0 *and standard deviation* $\sigma(h, h')$*.*

Then for all $N \in \mathcal{Z}_d$*, the random variable* $\frac{1}{(h')^{[\omega-1]}}(\Delta_{\mathbf{u}}^\omega \star \Gamma)(N) - f^{(\omega)}(Nh)$*, defined after the independent random variables* $\Gamma(N)$*, has expected value* $ES_\omega(f, h, h', \Gamma, \mathbf{u}, n)$ *and standard deviation* $\frac{\sigma(h,h')}{(h')^{[\omega]}} \left(\sum_{I \in \mathcal{Z}_d} (u(I))^2\right)^{\frac{1}{2}}$*.*

In other words, and roughly speaking, the global error is in this case statistically close to the sampling error.

Proof. From Equation (11) and Equation (13), for a fixed $N \in \mathcal{Z}_d$, the random variable $\frac{1}{(h')^{[\omega-1]}}(\Delta_{\mathbf{u}}^{\omega} \star \Gamma)(N) - f^{(\omega)}(Nh)$ is equal to the sum of the constant random variable $ES_\omega(f, h, h', \Gamma, \mathbf{u}, N)$ and the random variable defined by

$$EV_\omega(f, h, h', \Gamma, \mathbf{u}, N) = \frac{1}{(h')^{[\omega]}} \sum_{I \in \mathcal{Z}_d} u_I \varepsilon_{h,h'}(N - I).$$

By linearity of expected values, its expected value is $ES_\omega(f, h, h', \Gamma, \mathbf{u}, N)$, which shows the first part of the statement.

Since the random variables $\varepsilon_{h,h'}(N - I)$ are assumed to be independent, and the series $\sum_{I \in \mathcal{Z}_d} u_i$ is assumed to be absolutely convergent, the variance of $EV_\omega(f, h, h', \Gamma, \mathbf{u}, N)$ is equal to the sum for $I \in \mathcal{Z}_d$ of the variances of $\frac{u(I)}{(h')^{[\omega]}}\varepsilon_{h,h'}(N - I)$ which, for standard the deviation, yields $\left(\frac{|u(I)|}{(h')^{[\omega]}}\sigma(h, h')\right)^2$. $\square$

Remark 3.2 *Note that for a fixed mask, the values error (or its standard deviation) generally does not converge to zero when h' converges to 0. We shall propose below a way to make it tend to zero by adapting the mask to the digitization step (see Theorem 3.2 and Theorem 3.3 below).*

3.4 Skipping Masks: Cheap Multigrid Convergence

The idea is to adapt the mask to the step of digitization, in order to get $\frac{1}{(h')^{[\omega-\alpha]}}\left(\sum_{I \in \mathcal{Z}_d} |u(I)|\right)$ converging to zero along with h. For limiting the complexity of computation, we set the number of non zero coefficients of the mask fixed.

Definition 3.4 Let $L = (l_a)_{a=1,\dots,d} \in \left(\mathcal{R}_+^*\right)^d$ be a vector with d coordinates which are strictly positive elements of the base ring. We consider the following map:

$$\begin{cases} \mathcal{M} & \longrightarrow & \mathcal{M}/L = \prod_{a=1}^{d} (\mathcal{A}_a / l_a) \\ (t_a)_{a=1,\dots,d} & \longmapsto & (t_a / l_a)_{a=1,\dots,d} \end{cases}$$

Then, this map is an of analyzable spaces isomorphism, and is called called the *division by* L operation.

In the sequel of this section, $L = (l_a)_{a=1,\dots,d} \in (\mathcal{Z}_d)^d$ with each coordinate $l_a > 0_{\mathcal{M}}$ and l_a multiple of $1_{\mathcal{M}}$. We call the vector L the *skipping step* for our masks.

Definition 3.5 [Skipping Masks] Let $\mathbf{u}$ be an $\omega-$differentiation mask. The corresponding $\omega-$differentiation $L-$skipping mask $\mathbf{u}_L$ is defined by $\mathbf{u}_l(I) = \frac{1}{L^{[\omega]}} u(\frac{I}{L})$ if for all $a \in \{1, \dots, d\}$ the coordinate l_a divides $I(a)$, and equal to 0 in all other cases.

Remark 3.3 *For $K \in \mathbb{N}^d$, we have*

$$\sum_{I \in \mathcal{Z}_d} I^K \mathbf{u}_L(I) = L^{[K-\omega]} \sum_{I \in \mathcal{Z}_d} I^K \mathbf{u}(I).$$

Therefore, the mask $\mathbf{u}_L$ is an $\omega-$differentiation mask as well as $\mathbf{u}$.

We also have

$$\sum_{I\in\mathcal{Z}_d} |\mathbf{u}_L(I)| = \frac{1}{L^{[\omega]}} \sum_{I\in\mathcal{Z}_d} |\mathbf{u}(I)|.$$

This allows a convenient choice of L, depending on h, which yields a values error which converges to zero, either using Lemma 3.2 or Lemma 3.3. This is formalized in the following theorems, which specify the skipping step $L(h)$ to use as a function of the sampling step.

3.4.1 Uniform Multigrid Convergence with Uniform Noise or Bias

Theorem 3.2 *Let* $\mathbf{u}$ *be an* $\omega-$*differentiation mask with and* $\mathbf{u}_L$ *the corresponding* $\omega-$*differentiation* $L-$*skipping mask with skips of length* L*. Suppose that* $f : \mathcal{M} \longrightarrow \mathcal{M}'$ *is a* $C^{\omega+1}$ *(we remind the reader that* $(\omega+1)_a = \omega_a + 1$ *for all* a*) function. This means that the partial derivatives* $f^{(J)}$ *exist and are continuous for all* $0 \leq J \leq \omega+1$*, and* $f^{(\omega+\mathbb{1}_X)}$ *is bounded for any* $X \in \{1,\ldots,d\}$*.*

Let $\alpha \in]0,1]^d$*,* $K \in \mathcal{R}_+^*$ *and let* h *and* h' *be defined as at the beginning of Section 3.2. Suppose* $\Gamma : \mathcal{Z}_d \longrightarrow \mathcal{Z}_d$ *is such that* $|h'\Gamma(I) - f(hI)| \leq Kh^{[\alpha]}$ *for all* $I \in \mathcal{Z}_d$ *(which corresponds to our uniform noise/bias input values errors model).*
Then, using the skipping steps $L(h) = \left\lfloor h^{[-1+\frac{\omega\alpha}{\omega+1}]}\right\rfloor$*, we have:*

$$\left|\left(\frac{1}{(h')^{[\omega-1]}}\Delta_{\mathbf{u}_{L(h)}} \star \Gamma\right)(N) - f^{(\omega)}(Nh)\right| \in O(h^{[\frac{\alpha}{\omega+1}]})$$

Proof. First, we give an upper bound for the values error. From Lemma 3.2 and definitions, we have $\left|EV(f,h,h',\Gamma,\mathbf{u}_{L(h)},n)\right| \leq \frac{K}{(L(h))^{[\omega]}(h')^{[\omega-\alpha]}} \sum_{I\in\mathcal{Z}_d} |u(I)|$. If $L(h) = \left\lfloor h^{[-1+\frac{\alpha}{\omega+1}]}\right\rfloor$, it is easy to check that $\dfrac{1}{(L(h))^{[\omega]}(h')^{[\omega-\alpha]}} \leq \dfrac{h^{[\alpha-\frac{\omega\alpha}{\omega+1}]}}{1-h^{[\omega-\frac{\omega\alpha}{\omega+1}]}}$, which is $O(h^{[\frac{\alpha}{\omega+1}]})$.

We now turn to the sampling error. Let us consider the upper bounds provided by Lemma 3.1. We could use Equation (14) for a more explicit bound for the error, but we chose for the sake of simplicity to use Equation (15) instead. Also using Remark 3.3 we get:

$$ES(f,h,h',\Gamma,\mathbf{u}_{l(h)},n) = O\left(\sum_{a=1}^{d} h_a\,(L(h))_a^{(\omega_a+1)-\omega}\right)$$

Now, with $L(h) = \left\lfloor h^{[-1+\frac{\omega\alpha}{\omega+1}]}\right\rfloor$, we obtain $ES(f,h,h',\Gamma,\mathbf{u}_{l(h)},n) = O\left(h^{[\frac{\alpha}{\omega+1}]}\right)$ □

3.4.2 Stochastic Multigrid Convergence with Stochastic Noise

Theorem 3.3 *Let* $\mathbf{u}$ *be a* ω*-differentiation mask and let* $\mathbf{u}_L$ *be the corresponding* $\omega-$*differentiation* $L-$*skipping mask. Suppose that* $f : \mathcal{M} \longrightarrow \mathcal{M}'$ *is a* $C^{\omega+1}$ *function, and* $f^{(\omega+\mathbb{1}_X)}$ *is bounded for all* $X \subset \{1,\ldots,d\}$ *Let* $\alpha \in]0,1]^d$*, let* $K \in \mathcal{M}'$*, with* $K > 0_{\mathcal{M}'}$*, and let* h *and* h' *be defined as at the beginning of Section 3.2. Let* Γ *be a digitization of* f *with step* h *and a stochastic noise* $\varepsilon_{h,h'}$ *with expected value* $0_{\mathcal{M}'}$*, and standard deviation* $\sigma(h,h') \leq Kh^{[\alpha]}$*.*

Then for skipping steps $L(h) = \left\lfloor h^{[1-\frac{\alpha}{\omega+1}}\right\rfloor$*, and for* $N \in \mathcal{Z}_d$*, the random variable*

$$\left(\frac{1}{(h')^{[\omega-1]}}\Delta_{\mathbf{u}_{L(h)}} \star \Gamma\right)(N) - f^{(\omega)}(Nh)$$

has an expected value and a standard deviation which are $O(h^{[\frac{\alpha}{\omega+1}]})$*.*

The proof is similar to that of Theorem 3.2, but using Lemma 3.3 instead of Lemma 3.2.

4 Locally Analytical Functions

All along this section, we consider again the notations $\mathcal{R}$, $\mathcal{A}_a$, $\mathcal{M}$ and $\mathcal{M}'$ defined in Section 2. Moreover, $\mathcal{I}$ denotes a sub-algebra of $\mathcal{M}$ containing $\mathcal{Z}_d$ (typically $\mathcal{I} = \mathcal{Z}_d$ or $\mathcal{I} = \mathcal{M}$). At last Δ is a differentiation operator on functions from $\mathcal{I}$ to $\mathcal{M}'$.

We shall also use the following notions and notations concerning shift in vectors and functions, as well as division by positive vectors:

Definition 4.1 Let $\Phi : \mathcal{I} \longrightarrow \mathcal{M}'$ be a function. Given $L = (l_a)_{a=1,\dots,d} \in \mathcal{R}^d$ a vector with d coordinates which are elements of the base ring. We identify the vector L with the element $(l_a.1_{\mathcal{A}_a})_{a=1,\dots,d}$ of $\mathcal{M}$. We thus define $\tau^L(\Phi)$ the *$L-$shift of* Φ which to $T \in \mathcal{I}$ associates

$$\left(\tau^L(\Phi)\right)(T) = \Phi(T + L)$$

We remind the reader of Definition 3.4, in which the definition of the (coordinate by coordinate) division by a vector $L = (l_a)_{a=1,\dots,d} \in \left(\mathcal{R}_+^*\right)^d$ is presented. In the sequel of this section, $L = (l_a)_{a=1,\dots,d} \in \left(\mathcal{R}_+^*\right)^d$ denotes a vector with d coordinates which are strictly positive elements of the base ring.

Definition 4.2 For $a \in 1, \dots, d$, let Δ_a be a differentiation operator over the analyzable space $\mathcal{A}_a$, with values in $\mathcal{M}'$. For any function $f : \mathcal{M} \longrightarrow \mathcal{M}'$, if for $T = (t_a)_{a=1,\dots,d}$ the function $f_{a,T} : \mathcal{A}_a \longrightarrow \mathcal{M}'$ which to $t \in \mathcal{A}_a$ associates $f(T^{(a,t-t_a)}$ is differentiable relatively to Δ_a, we denote

$$\frac{\partial}{\partial t_a}(T) = \left(\Delta_a\left(f_{a,T}\right)\right)(T)$$

Moreover, this value is called the *partial derivative of f with respect to (the a^{th} coordinate) t_a at the point T.*

4.1 Definition of Differential $B-$Splines Families

Definition 4.3 Let us consider a family $\mathcal{D} = (D_{I,S,P,R,L})$ of functions from $\mathcal{I}/L$ to $\mathcal{M}'/L^R$, where, roughly speaking,

- $S = (s_a)_{a=1,\dots,d} \in \mathcal{Z}_d$ is a shift factor, through which the parameter $T = (t_a)_{a=1,\dots,d}$ of functions is translated.
- $L = (l_a)_{a=1,\dots,d} \in \left(\mathcal{R}_+^*\right)^d$ denotes a vector with d coordinates which are strictly positive elements of the base ring, and determines a partition of $\mathcal{I}$ into intervals $[S.L, (S+1)L[$.
- $R = (r_a)_{a=1,\dots,d} \in \mathbb{N}^d$ is a blunder order, or smoothing order, which determines the regularity of elements of $\mathcal{D}$, as functions on $\mathcal{I}$.
- $P = (p_a)_{a=1,\dots,d} \in \mathbb{N}^d$ denotes the *primitive order*, which represents the number of times the primitive operator was applied, in the respective dimensions, relative to the differentiation operator Δ, on the corresponding function with $P = 0$.

- $\delta_R = (\delta_{R,a})_{a=1,\dots,d}$ denotes the dimension of the space $\mathcal{D} = (D_{I,0,0,R,L})_{0\leq I\leq \delta_R}$, of the functions in the family $\mathcal{D}$ for a fixed L. The index L is omitted in the notation δ_R because, in the families we present in this paper, the dimension δ_R does not depend on L.

Now defining precisely, using Notation 2.1, we say that the family $\mathcal{D}$ is a *Differential B−spline Family* of functions with respect to Δ if and only if it satisfies the four following properties, valid for all $S \in \mathbb{Z}_d$, $P \in \mathbb{N}^d$, $R \in \mathbb{N}^d$ and any vector $L \in \left(\mathcal{R}_+^*\right)^d$ with d coordinates which are strictly positive:

1. Differential Property: for $T = (t_a)_{a=1,\dots,d}$, we have
$$\frac{\partial}{\partial t_a}(D_{I,S,P^{(a,1)},R,L})(T) = (p_a+1)D_{I,S,P,R,L}(T)$$
2. Commutation with Finite Differences Property:
$$D_{I,S,P,R^{(a,1)},L} = \frac{1}{l_a(p_a+1)(r_a+1)}(D_{I,S,P^{(a,1)},R,L} - D_{I,S^{(a,1)},P^{(a,1)},R,L})$$
3. Shift Property:
$$D_{I,S^{(a,-1)},P,R,L} = \tau^{L(a,l_a)}\left(D_{I,S,P,R,L}\right)$$
4. Partition of Unity Property: For all $T \in \mathcal{I}$ and for $P = 0$,
$$\sum_{S\in\mathcal{Z}_d}\sum_{0\leq I\leq\delta_R} D_{I,S,0,R,L}(T) = \frac{1_{\mathcal{M}'}}{L^{[R]}}$$

4.2 Generic Construction from Partitions of Unity

Definition 4.4 A function $F : \mathcal{I} \longrightarrow \mathcal{M}'$ is said to be *eventually zero when the coordinates tend to* $-\infty$ if there exists $U \in \mathcal{I}$ such that $F(T) = 0_{\mathcal{M}'}$ for $T \leq U$.

Let us consider a family $\mathcal{D} = (D_{I,0,0,0,1})$ of functions from $\mathcal{I}$ to $\mathcal{M}'$ such that: For all $T \in \mathcal{I}$, we have the *partition of unity property*:
$$\sum_{S\in\mathcal{Z}_d}\sum_{0\leq I\leq\delta_O} D_{I,0,0,0,1}(T) = 1_{\mathcal{M}'}$$
We extend the family $\mathcal{D}$ to a complete family (also denoted by $\mathcal{D} = (D_{I,S,P,R,L})$) as follows.

1. For $L = (l_a)_{a=1,\dots,d} \in \left(\mathcal{R}_+^*\right)^d$ a vector with d coordinates which are strictly positive elements of the base ring, we set:
$$D_{I,S,0,0,L}(T) = \frac{1}{L^{[R]}}D_{I,0,0,0,1}(\frac{T}{L} - S)$$
2. We define by induction on $P = (p_a)_{a=1,\dots,d} \in \mathbb{N}^d$ the function $D_{I,S,P,0,L}$, by setting for $T = (t_a)_{a=1,\dots,d}$:
$$D_{I,S,P^{(a,1)},0,L}(T) = \int_{-\infty}^{t_a} D_{I,S,P,0,L}\left(T^{(a,u-t_a)}\right) du$$
Note that the integral is well defined for a function which is eventually zero when the coordinates tend to $-\infty$. Furthermore, if $D_{I,S,0,0,L}$ is eventually zero when the coordinates tend to $-\infty$, then so is $D_{I,S,P,0,L}$ for any $P \in \mathbb{N}^d$.

3. At last we define by induction on $R = (r_a)_{a=1,\dots,d} \in \mathbb{N}^d$ the function $D_{I,S,P,R,L}$, by setting:

$$D_{I,S,P,R^{(a,1)},L} = \frac{1}{l_a(p_a+1)(r_a+1)}(D_{I,S,P^{(a,1)},R,L} - D_{I,S^{(a,1)},P^{(a,1)},R,L})$$

Then, we have the following result, which follows from the definition by a straightforward induction:

Proposition 4.1 *The family $\mathcal{D} = (D_{I,S,P,R,L})$ is a Differential $B-$spline family.*

4.3 Generalized Cox-de-Boor Formula

Theorem 4.1 (Generalized Cox-de-Boor Relation) *Let $\mathcal{D} = (D_{I,S,P,R,L}))$ is be a differential $B-$spline family. For all $S \in \mathbb{Z}_d$, $R \in \mathbb{N}^d$, for any vector $L \in \left(\mathcal{R}_+^*\right)^d$ with d coordinates which are strictly positive, for any $a \in \{1,\dots,d\}$ and for any $T \in \mathcal{I}$, we have:*

$$D_{I,S,0,R^{(a,1)},L}(T) = \frac{t_a - s_a}{l_a(r_a+1)} D_{I,S,0,R,L}(T) + \frac{s_a + r_a - t_a}{l_a(r_a+1)} D_{I,S^{(a,1)},0,R,L}(T)$$

Proof. We prove the result by induction on R. For $R = 0$ and any $P\mathbb{N}^d$, we have

$$\begin{array}{rcl}
D_{I,S,P,R^{(a,1)},L}(T) & = & \frac{1}{l_a(p_a+1)(r_a+1)}\left(D_{I,S,P^{(a,1)},R,L}(T) - D_{I,S^{(a,1)},P^{(a,1)},R,L}(T)\right) \\
& = & \frac{1}{l_a(r_a+1)}\left(\int_{-\infty}^{t_a} D_{I,S,P,R,L}\left(T^{(a,u_a-t_a)}\right) du_a - \int_{-\infty}^{t_a} D_{I,S^{(a,1)},P,R,L}\left(T^{(a,u_a-t_a)}\right) du_a\right) \\
& = & \frac{1}{l_a(r_a+1)}\left(\left[(u_a - s_a)D_{I,S,P,R,L}(T^{(a,u_a-t_a)})\right]_{u_a=-\infty}^{u_a=t_a}\right. \\
& & \qquad \left. - \left[(u_a - r_a - s_a)D_{I,S^{(a,1)},P,R,L}(T^{(a,u_a-t_a)})\right]_{u_a=-\infty}^{u_a=t_a}\right) \\
& & -\frac{1}{l_a(r_a+1)}\left(\int_{-\infty}^{t_a}(u_a - s_a)\frac{\partial}{\partial u_a}(D_{I,S,P,R,L})\left(T^{(a,u_a-t_a)}\right) du_a\right. \\
& & \qquad \left. - \int_{-\infty}^{t_a}(u_a - r_a - s_a)\frac{\partial}{\partial u_a}\left(D_{I,S^{(a,1)},P,R,L}\left(T^{(a,u_a-t_a)}\right)\right) du_a\right) \\
& = & \frac{t_a-s_a}{l_a(r_a+1)} D_{I,S,P,R,L}(T) + \frac{s_a-r_a-t_a}{l_a(r_a+1)} D_{I,S^{(a,1)},P,R,L}(T) \\
& & +\frac{1}{l_a(r_a+1)} \int_{-\infty}^{t_a}\left((u_a - s_a)D_{I,S,P^{(a,-1)},R,L}(T^{(a,u_a-t_a)})\right. \\
& & \qquad \left. + (s_a + r_a - u_a)D_{I,S^{(a,1)},P^{(a,-1)},R,L}(T^{(a,u_a-t_a)})\right) du_a
\end{array}$$

Now, for $P = 0$ as in our statement, we have $D_{I,S,P^{(a,-1)},R,L} \equiv 0_{\mathcal{M}'}$ and $D_{I,S^{(a,1)},P^{(a,-1)},R,L} \equiv 0_{\mathcal{M}'}$ due to the differential property in Definition 4.3, which completes the proof. □

4.4 Generalized Power Series and Analytical Functions

Note on the draft version. The remainder of this section is somewhat sketch for lack of time. The final version of this draft ought to contain more about generalized power series, especially as solutions to linear partial differential equations.

Let $\mathbf{D} = (D_{I,S,P,R,L})$ be a differential $B-$spline family with respect to a differentiation operator Δ over $\mathcal{M}$, which is obtained by tensor product of differentiation operators Δ_a for $a = 1,\dots,d$. For $\omega = (\omega_a)_{a=1,\dots,d} \in \mathbb{N}^d$, we denote by $\Delta^{(\omega)}$ the partial derivative of order ω_a using Δ_a on $\mathcal{A}_a$.

For the sake of simplicity, we assume that the functions $D_{I,0,0,R,L}$ have bounded support, namely that $\text{supp}(D_{I,0,0,R,L}) \subset [-m(R), m(R)]$ for some positive element $m(R) \in \mathcal{I}$. We also assume that $D_{I,0,0,R,L}(T) > 0_{\mathcal{M}'}$ for all $T \in \mathcal{I}$, which implies, from the partition of unity property, that $||D_{I,0,0,R,L}||_\infty \leq \frac{1}{L^{[R]}}$.

Note, however, that the content of this paper regarding analytical functions and their applications might work also for such functions families of functions with rapidly decreasing derivatives of all orders, such as constructed as in Section 4.2 using partitions of unity, as well as for some families of non positive functions.

Lemma 4.1 *For $P \in \mathbb{N}^d$ and $b > 0_{\mathcal{I}}$, the suppremum M of $||D_{I,S,P,R,L}(T)||_\infty$ for x element of an interval $[-b, b] \subset \mathcal{I}$ is less that or equal to*

$$(|S| + 2m(R))^{[P]}$$

Proof. Let T be an element of an interval $[A, B] \subset \mathcal{I}$. Since $||D_{I,0,0,R,L}||_\infty \leq 1$, the result is true for $P = 0$. We then show the result by induction on P. Assuming it is true for P, we see that $||D_{I,S,P^{(a,+1)},R,L}(T)||_\infty = ||\int_S^T D_{I,S,P,R,L}(U)||_\infty dU \leq \int_0^{|S|+2m(R)} (|S| + 2m(R))^{[P]} dU = (|s| + 2m(R))^{[P^{(a,+1)}]}$. □

Lemma 4.2 *Let $\mathbf{c} = (c_{I,S,P})$, for $0 \leq I \leq \delta_R$, $S \in \mathcal{Z}_d$, and $P \in \mathbb{N}^d$ be a family of elements of $\mathcal{R}$ such that*

$$\sum_{S\in\mathcal{Z}_d} \sum_{0\leq P\leq N} \sum_{0\leq I\leq\delta_R} ||c_{I,S,P}||_A (|S| + 2m(R))^P \tag{16}$$

is absolutely convergent when all coordinates of $N \in \mathbb{N}^d$ tend $+\infty$. Then for any $R \in \mathbb{N}$ and $S \in \mathcal{Z}_d$, the sum

$$\mathcal{S}_{\mathbf{c},\mathbf{D},N}(T) = \sum_{S\in\mathcal{Z}_d} \sum_{0\ \leq P\leq N} \sum_{0\leq I\leq\delta_R} c_{I,S,P} D_{I,S,P,R,L}(T) \tag{17}$$

also converges when all coordinated of $N \to +\infty$.

Definition 4.5 Under the assumptions of Lemma 4.2, the coefficients $\mathbf{c} = (c_{I,S,P})$ are said to *define a convergent generalized power series relative to* Δ *and* $\mathbf{D}$ Moreover, the limit for $n \to +\infty$ for the sums considered in Equation (17) is called the sum of the *generalized power series relative to* Δ *and* $\mathbf{D}$ *with coefficients* $(c_{I,S,P})$*, with scaling factor* L*, with blending order* R

$$\mathcal{S}_{\mathbf{c},\mathbf{D}}(T) = \sum_{S\in\mathcal{Z}_d} \sum_{0\leq P} \sum_{0\leq I\leq\delta_R} c_{I,S,P} D_{I,S,P,R,L}(T) \tag{18}$$

Definition 4.6 A function f from $\mathcal{I}$ to $\mathcal{M}'$ is called a *generalized analytical function relative to* Δ *and* $\mathbf{D}$ if is can be expressed as a generalized power series relative to Δ and $\mathbf{D}$ for some coefficients $(c_{I,S,P})$, with scaling factor L, with blending order R

Proposition 4.2 (Differentiation of Generalized Analytical Functions) *Let $R \geq 1_{\mathbb{N}}^d$, let $\mathbf{c} = (c_{I,S,P})$, for $0 \leq I \leq \delta_R$, $S \in \mathcal{Z}_d$, and $P \in \mathbb{N}^d$ be a family of elements of $\mathcal{R}$ which define a convergent generalized power series relative to Δ and $\mathbf{D}$. Then the function $\mathcal{S}_{\mathbf{c},\mathbf{D}}$ is differentiable for Δ (i.e. $\Delta(\mathcal{S}_{\mathbf{c},\mathbf{D}})$ exists) and we have:*

$$\Delta_a\left(\mathcal{S}_{\mathbf{c},\mathbf{D}}\right)(T) = \sum_{S\in\mathcal{Z}_d} \sum_{0\leq P=0} \sum_{0\leq I\leq\delta_R} c_{I,S,P^{(a,+1)}} D_{I,S,P,R,L}(T) \tag{19}$$

Proof. From the differential property of the differential $B-$spline family $\mathbf{D}$, we get for $N \in \mathbb{N}^d$ that the sum $\mathcal{S}_{\mathbf{c},\mathbf{D},N}(T)$ defined in Equation (17), as a function of $T \in \mathcal{I}$, is derivable for Δ and its derivative is

$$\Delta_a\left(\mathcal{S}_{\mathbf{c},\mathbf{D},N}\right)(T) = \sum_{S\in\mathcal{Z}_d}\sum_{0\le P\le N}\sum_{0\le I\le\delta_R} c_{i,s,p+1}D_{I,S,P,R,L}(T)$$

Furthermore, we see that the coefficients $\mathbf{a}' = (c_{i,s,p+1})$ for $0 \le I \le \delta_R$, $S \in \mathcal{Z}_d$, and $P \in \mathbb{N}^d$ defines a convergent generalized power series relative to Δ and $\mathbf{D}$, that is, the series $\sum_{D\in\mathcal{Z}_d}\sum_{0\le P\le N}\sum_{0\le I\le\delta_R} ||c_{I,S,P^{(a,1)}}||_\infty(|s|+2m(R))^{[P]}$ converges. Hence the series of Equation (19) above converges when all coordinates of N tend to $\to +\infty$, and, using the continuity of the operator Δ, by taking the limit when $N \to +\infty$ we get our result. □

By an immediate induction on Proposition 4.2, we get the following

Theorem 4.2 *Let $R \ge 1^d_{\mathbb{N}}$, Let $\mathbf{c} = (c_{I,S,P})$, for $0 \le I \le \delta_R$, $S \in \mathcal{Z}_d$, and $P \in \mathbb{N}^d$ be a family of elements of $\mathcal{R}$ which define a convergent generalized power series relative to Δ and $\mathbf{D}$. Then, for any $\omega \in \mathcal{N}^d$, the function $\mathcal{S}_{\mathbf{c},\mathbf{D}}$ is $\omega-$differentiable for Δ (i.e. $\Delta^\omega(\mathcal{S}_{\mathbf{c},\mathbf{D}})$ exists) and we have:*

$$\Delta^\omega(\mathcal{S}_{\mathbf{c},\mathbf{D}})(T) = \sum_{s\in\mathbb{Z}}\sum_{p=0}^{+\infty}\sum_{0\le I\le\delta_R} c_{I,S,P+\omega}D_{I,S,P,R,L}(T) \tag{20}$$

4.5 Solutions of Linear Differential Equations

Le us consider a linear partial differential equation of the form:

$$\sum_{0\le J\le K}\alpha_J(T)\left(\Delta^{(J)}(f)\right)(T) = 0 \tag{21}$$

where $K \in \mathbb{N}^d$. Let us look for generalized analytical functions which are solutions.

So, as in Section 4.4, let $\mathbf{D} = (D_{I,S,P,R,L})$ be a differential $B-$spline family with respect to a differentiation operator Δ over $\mathcal{M}$. We assume, as has been proven for some differential $B-$spline families in section 4.4, that Definition 4.5 holds, as well as Theorem 4.2.

Let $\mathbf{c} = (c_{I,S,P})$, for some $R \ge 1_{\mathbb{N}^d}$, let $0 \le I \le \delta_R$, $S \in \mathcal{Z}_d$, and $P \in \mathbb{N}^d$ be a family of elements of $\mathcal{R}$ which define a convergent generalized power series relative to Δ and $\mathbf{D}$. From Theorem 4.2, it is sufficient that the coefficients $(c_{I,S,P})$ satisfy the following linear equations, for every $T = (t_a)_{a=1,\dots,d}$:

$$\sum_{0\le J\le K}\alpha_j(T)c_{I,S,P+J}D_{I,S,P,R,L}(T) = 0 \tag{22}$$

Example 4.1 *In the one dimensional case ($d = 1$) real case $\mathcal{R} = \mathcal{M} = \mathcal{M}' = \mathbb{R}$. Let us consider the equation $\Delta(f) = f$ (which is classically solved to get the exponential function $T \to e^T$). Equation (22) yields:*

$$c_{I,S,P+1} = c_{I,S,P}$$

We therefore get the following family of solutions, for any given $P \in \mathbb{N}$ and $L > 0$:

$$\exp_{\mathbf{N}}(T) \overset{\text{def}}{=} \sum_{S\in\mathbb{Z}}\sum_{P=0}^{+\infty} c_{0,S,0}N_{0,S,P,R,L}(T)$$

Where $(c_{0,S,0})_{S\in\mathbb{Z}}$ *is an arbitrary sequence. This example was implemented, to get the results presented on Figure 1.*

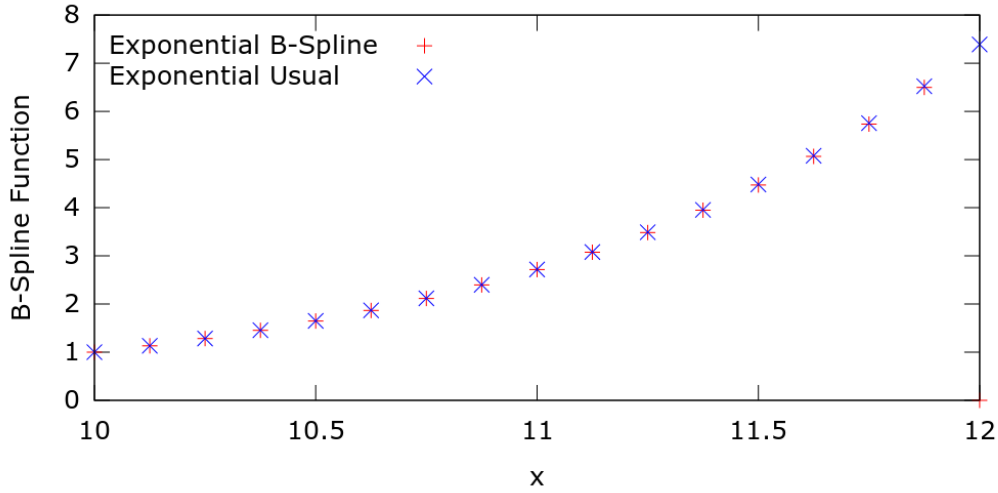

Figure 1: The results of a partial sum (for a finite N) of $B-$splines obtained as in Example 4.1. The graph superimposes perfectly with the usual exponential e^x.

The final version of the paper ought to provide more about linear partial differential equations, as how to choose solutions of the linear equations in Equation (22) to obtain an integer only and drift-free solution.

5 Bernstein-Based Differential $B-$Splines

5.1 Bézier Functions and Bernstein polynomials Basics

Note that the following definition uses multidimensional binomial coefficients and exponents following Notation 2.1, as well as polynomial functions introduced in Section 2.3.

Definition 5.1 Let $R \in \mathbb{N}^d$. For $I \in \mathbb{N}^d$ with $0 \le I \le R$, we consider the $\mathcal{M}'-$valued polynomial with degree R which defines for $T \in \mathcal{I}$ the element

$$B_{I,R}(T) = \binom{R}{I}(T^I 1_{\mathcal{M}'})\,(1_{\mathcal{M}'} - T1_{\mathcal{M}'})^{R-I}$$

These (in the framework of $\mathbb{R}-$vector spaces well-known) polynomials are called the *Bernstein polynomials with degree* R from $\mathcal{I}$ to $\mathcal{M}'$.

In the sequel, unless otherwise specified, we shall say Bernstein polynomials or Bernstein functions as a shorthand for Bernstein polynomials from $\mathcal{I}$ to $\mathcal{M}'$. The Bernstein polynomials with degree R constitute, as formal polynomials, a basis of the vector space of polynomials with degree less than or equal to R. We shall often omit the $1_{\mathcal{M}'}$ factors if no ambiguity can arise, thus writing:

$$B_{I,R}(T) = \binom{R}{I}T^I\,(1_{\mathcal{M}'} - T)^{R-I}$$

Remark 5.1 (Partition of Unity Property) *By developing* $(T+(1-T))^R$ *we obtain* $\sum_{0\leq I\leq R} B_{I,R}(T) =$ 1 *for all* $T \in\in \mathcal{I}$

From the Pascal formula (remark 2.4) for binomial coefficients, we derive a similar formula about Bernstein polynomials:

Proposition 5.1

$$B_{I,R}(T) = (1_{\mathcal{M}'} - t_a 1_{\mathcal{M}'})B_{I,R^{(a,-1)}}(T) + (t_a 1_{\mathcal{M}'})B_{I^{(a,-1)},R^{(a,-1)}}(T) \tag{23}$$

By omitting the unit $1_{\mathcal{M}'}$, we can equivalently write:

$$B_{I,R}(T) = (1_{\mathcal{A}_a} - t_a)B_{I,R^{(a,-1)}}(T) + t_a B_{I^{(a,-1)},R^{(a,-1)}}(T) \tag{24}$$

Proof.

$$\begin{array}{rcl} B_{I,R}(T) & = & \binom{R}{I} T^I \left(1_{\mathcal{M}'} - T\right)^{R-I} \\ & = & \left[\binom{R^{(a,-1)}}{I^{(a,-1)}} + \binom{R^{(a,-1)}}{I}\right] T^I \left(1_{\mathcal{M}'} - T\right)^{R-I} \\ & = & t_a \binom{R^{(a,-1)}}{I^{(a,-1)}} T^{I^{(a,-1)}} \left(1_{\mathcal{M}'} - T\right)^{R^{(a,-1)}-I^{(a,-1)}} + (1_{\mathcal{M}'} - t_a)\binom{R^{(a,-1)}}{I} T^I \left(1_{\mathcal{M}'} - T\right)^{R^{(a,-1)}-I} \\ & = & (1_{\mathcal{M}'} - t_a 1_{\mathcal{M}'})B_{I,R^{(a,-1)}}(T) + (t_a 1_{\mathcal{M}'})B_{I^{(a,-1)},R^{(a,-1)}}(T) \end{array}$$

□

Definition 5.2 Let $R \in \mathbb{N}^d$. Let $\mathbf{P} = (P_I)_{0\leq I\leq R}$ be a multi-sequence of points in $\mathcal{M}'$. We define, for $T = (t_a)_{a=1,\ldots,d} \in \mathcal{I}$, the image of T under the *Bézier function* $B_{\mathbf{P}} : \mathcal{I} \longrightarrow \mathcal{M}'$ with control points $\mathbf{P}$ by

$$B_{\mathbf{P}}(T) = \sum_{0\leq I\leq R} P_I B_{I,R}(T)$$

Now, if we wan to compute partial differentials for Bernstein polynomials, we consider the (in $\mathbb{R}^d$ classical) formula for Bernstein polynomials are concerned. For $T = (t_a)_{a=1,\ldots,d}$, we have:

$$\frac{\partial}{\partial t_a} B_{I,R}(T) = r_a (B_{I^{(a,-1)},R^{(a,-1)}}(T) - B_{I,R^{(a,-1)}}(T)) \tag{25}$$

Proposition 5.2 *As in the usual case of Bézier functions over* $\mathbb{R}$*, the partial derivative of a Bézier function from* $\mathcal{I}$ *to* $\mathcal{M}'$ $B_{\mathbf{P}} : \mathcal{I} \longrightarrow \mathcal{M}'$ *with control points* $(P_I)_{0\leq I\leq R}$ *can be expressed as follows, denoting* $T = (t_a)_{a=1,\ldots,d}$*:*

$$\frac{\partial}{\partial t_a} B_{\mathbf{P}}(T) = r_a \sum_{0\leq I\leq R^{(a,-1)}} (P_{I^{(a,1)}} - P_I) B_{I,R^{(a,-1)}}(T)$$

which is the Bézier function with control points $(P'_I)_{0\leq I\leq R^{(a,-1)}}$*, where* $P'_I = r_a(P_{I^{(a,1)}} - P_I)$*.*

Proof.

$$\begin{array}{rcl} \frac{\partial}{\partial t_a} B_{\mathbf{P}}(T) & = & r_a \sum_{0\leq I\leq R} P_I \left(B_{I^{(a,-1)},R^{(a,-1)}}(T) - B_{I,R^{(a,-1)}}(T)\right) \\ & = & r_a \left(\sum_{0\leq I\leq R} P_I\, B_{I^{(a,-1)},R^{(a,-1)}}(T) - \sum_{0\leq I\leq R} P_I\, B_{I,R^{(a,-1)}}(T)\right) \\ & = & r_a \sum_{0\leq I\leq R^{(a,-1)}} (P_{I^{(a,1)}} - P_I) B_{I,R^{(a,-1)}}(T) \end{array}$$

□

5.2 Scaled Bézier Function Associated to a Sequence

Notation 5.1 *Let u be an element of $\mathcal{M}/L$ (see Definition 3.4). We consider the* floor of u, *denoted by $\lfloor u \rfloor$, which is the greatest element (considering the coordinate by coordinate partial order on $\mathcal{M}$) of $\mathcal{Z}_d$ such that $L\left(\frac{\lfloor u \rfloor}{L}\right)$ is less than or equal to $L\left(\frac{u}{L}\right)$ in $\mathcal{M}/L$ (considering the coordinate by coordinate partial order on $\mathcal{M}/L$).*

Definition 5.3 Let $L \in \left(\mathcal{R}_+^*\right)^d$ be a vector with d coordinates which are strictly positive elements of the base ring. Let $S \in \mathcal{Z}_d$. For $R \in \mathbb{N}^d$ and $I \in \mathbb{N}^d$ with $0 \leq I \leq R$, we introduce the *$S-$shifted $L-$scaled Bernstein polynomials with degree R*, with values in $\mathcal{M}'/L^R$, by:

$$B_{I,S,R,L}(T) = \begin{cases} B_{I,R}\left(\frac{T}{L} - S\right) \text{ if } \frac{T}{L} \in [S, S+1_{\mathcal{M}}[\\ 0 \text{ otherwise.} \end{cases}$$

In other words, the value $B_{I,S,R,L}(T)$ can be non-zero only for $S = \left\lfloor \frac{T}{L} \right\rfloor$. Using the caracteristics function of an interval, we can also write $B_{I,S,R,L}(T) = B_{I,R}\left(\frac{T}{L} - S\right) \mathbb{1}_{[s,\, s+1\ \ _M[}$.

Remark 5.2 (Shift Property)

$$\left(\tau^{L(a,l_a)}\left(B_{I,S,R,L}\right)\right)(T) = B_{I,S,R,L}(T^{(a,l_a)}) = B_{I,S^{(a,-1)},R,L}(T)$$

In the remainder of this section $(\Gamma(S))_{S\in\mathcal{Z}_d}$ is a multi-sequence with values in $\mathcal{M}'$, and $L = (l_a)_{a=1,\dots,d}$ is an element of $\left(\mathcal{R}_+^*\right)^d$.

Definition 5.4 For $R \in \mathbb{N}^d$, the *$L-$scaled (piecewise) Bézier function with degree R associated to* Γ is defined for $T \in \mathcal{M}$ by:

$$\left(\mathcal{B}_{L,R}^{(0)}(\Gamma)\right)(T) = \sum_{S\in\mathcal{Z}_d}\sum_{0\leq I\leq R} \Gamma\left(L(I-S)\right) B_{I,S,R,L}(T)$$

Note that in the previous definition, due to the definition of $B_{I,S,R,L}$, for a given value of I, only one value of S (namely $S = \left\lfloor \frac{T}{L} \right\rfloor$) contributes to the double sum $\left(\mathcal{B}_{L,R}^{(0)}(\Gamma)\right)(T)$, so that, in fact, at most $(|R| + d)$ terms are non-zero for a given T.

Proposition 5.3 (Commutation with the Shift) *For $R \in \mathbb{N}^d$ and $I \in \mathbb{N}^d$ with $0 \leq I \leq R$, for $T = (t_a)_{a=1,\dots,d} \in \mathcal{M}$, we have:*

$$\left(\mathcal{B}_{L,R}^{(0)}(\Gamma)\right)(\tau^{L(a,l_a)}(T)) = \left(\mathcal{B}_{L,R}^{(0)}(\tau^{-L(a,l_a)}(\Gamma))\right)(T)$$

Proof.

$$\begin{aligned}
\left(\mathcal{B}_{L,R}^{(0)}(\Gamma)\right)(\tau^{L(a,l)}(T)) &= \textstyle\sum_{S\in\mathcal{Z}_d}\sum_{0\leq I\leq R} \Gamma\left(L(I-S)\right) B_{I,S,R,L}(T^{(a,l_a)}) \\
&= \textstyle\sum_{S\in\mathcal{Z}_d}\sum_{0\leq I\leq R} \Gamma\left(L(I-S)\right) B_{I,S^{(a,-1)},R,L}(T) \\
&= \textstyle\sum_{S\in\mathcal{Z}_d}\sum_{0\leq I\leq R} \Gamma\left(L(I-S^{(a,1)})\right) B_{I,S,R,L}(T) \\
&= \textstyle\sum_{S\in\mathcal{Z}_d}\sum_{0\leq I\leq R} \Gamma\left(L(S-I) - 0^{(a,l_a)})\right) B_{I,S,R,L}(T) \\
&= \left(\mathcal{B}_{L,R}^{(0)}(\tau^{-L(a,l_a)}(\Gamma))\right)(T)
\end{aligned}$$

□

Proposition 5.4 (De Casteljau Property on Sequences) *Using the elements $L(a,j)$ for $a = 1,\dots,d$ and $j \in \mathcal{A}_a$, as well as $R^{(a,-1)} \in \mathbb{N}^d$, defined in Notation 2.1, we have for $T = (t_a)_{a=1,\dots,d}$:*

$$\left(\mathcal{B}^{(0)}_{L,R}(\Gamma)\right)(T) = \left(1 - \left(\frac{t_a}{l_a} - \left\lfloor\frac{t_a}{l_a}\right\rfloor\right)\right)\left(\mathcal{B}^{(0)}_{L,R^{(a,-1)}}(\Gamma)\right)(T) + \left(\frac{t_a}{l_a} - \left\lfloor\frac{t_a}{l_a}\right\rfloor\right)\left(\mathcal{B}^{(0)}_{L,R^{(a,-1)}}(\tau^{L(a,l_a)}(\Gamma))\right)(T)$$

In the equation, we omitted $1_{\mathcal{M}'}$ when multiplying $\mathcal{M}'-$valued polynomials by $\left(\frac{t_a}{l_a} - \left\lfloor\frac{t_a}{l_a}\right\rfloor\right)1_{\mathcal{M}}$ or $\left(1_{\mathcal{A}_a} - \left(\frac{t_a}{l_a} - \left\lfloor\frac{t_a}{l_a}\right\rfloor\right)\right)1_{\mathcal{M}}$, seen as degree zero monomials (Definition 2.10). This notation is also valid for $R = 0$ if we use the convention that $\mathcal{B}^{(0)}_{L,R^{(a,j)}} = 0$ if $r_a + j < 0$.

Proof.

$$\begin{array}{rcl}\left(\mathcal{B}^{(0)}_{L,R}(\Gamma)\right)(T) & = & \sum_{S\in\mathcal{Z}_d}\sum_{0\le I\le R}\Gamma\left(L(I-S)\right)B_{I,R}(\frac{T}{L}-S)\mathbb{1}_{[S,S+1_{\mathcal{M}}[}\left(\frac{T}{L}\right)\\ & = & \sum_{S\in\mathcal{Z}_d}\sum_{0\le I\le R}\ \Gamma\left(L(I-S)\right)\\ & & .\left[\left(1_{\mathcal{A}_a} - \left(\frac{t_a}{l_a} - s_a\right)\right)B_{I,R^{(a,-1)}}(\frac{T}{L}-S)\right.\\ & & \left.+\left(\frac{t_a}{l_a} - s_a\right)B_{I^{(a,-1)},R^{(a,-1)}}(\frac{T}{L}-S)\right].\mathbb{1}_{[S,S+1[}\left(\frac{T}{L}\right)\end{array}$$

The last equality follows from Equation 24. Now, taking into account that the only value of S for which $\frac{T}{L} \in [S, S+1[$, which implies that $\frac{t_a}{l_a} - s_a = \frac{t_a}{l_a} - \left\lfloor\frac{t_a}{l_a}\right\rfloor$, and then by changing the index I to $I^{(a,-1)}$ in the sum, we get:

$$\begin{array}{rcl}\left(\mathcal{B}^{(0)}_{L,R}(\Gamma)\right)(T) & = & \left(1_{\mathcal{A}_a} - \frac{t_a}{l_a} + \left\lfloor\frac{t_a}{l_a}\right\rfloor)\right)\sum_{S\in\mathcal{Z}_d}\sum_{0\le I\le R}\Gamma\left(L(I-S)\right)B_{I,S,R^{(a,-1)},L}(T)\\ & & +\left(\frac{t_a}{l_a} - \left\lfloor\frac{t_a}{l_a}\right\rfloor\right)\sum_{S\in\mathcal{Z}_d}\sum_{0^{(a,-1)}\le I\le R^{(a,-1)}}\Gamma\left(L(I^{(a,1)}-S)\right)B_{I,S,R^{(a,-1)},L}(T)\\ & = & \left(1_{\mathcal{A}_a} - \frac{t_a}{l_a} + \left\lfloor\frac{t_a}{l_a}\right\rfloor)\right)\left(\mathcal{B}^{(0)}_{L,R^{(a,-1)}}(\Gamma)\right)(T) + \left(\frac{t_a}{l_a} - \left\lfloor\frac{t_a}{l_a}\right\rfloor\right)\left(\mathcal{B}^{(0)}_{L,R^{(a,-1)}}(\tau^{(a,l_a)}(\Gamma))\right)(T)\end{array}$$

The indices $I = 0^{(a,-1)}$ and $I = R$ yielding a zero term because out of range for the Bernstein polynomials. □

Now, we derive the following from Proposition 5.3 and Proposition 5.4:

Proposition 5.5 (De Casteljau Property on Functions) *We have for $T = (t_a)_{a=1,\dots,d}$:*

$$\left(\mathcal{B}^{(0)}_{L,R}(\Gamma)\right)(T) = \left(1 - \left(\frac{t_a}{l_a} - \left\lfloor\frac{t_a}{l_a}\right\rfloor\right)\right)\left(\mathcal{B}^{(0)}_{L,R^{(a,-1)}}(\Gamma)\right)(T) + \left(\frac{t_a}{l_a} - \left\lfloor\frac{t_a}{l_a}\right\rfloor\right)\left(\tau^{-L(a,l_a)}(\mathcal{B}^{(0)}_{L,R^{(a,-1)}}(\Gamma))\right)(T)$$

5.2.1 Derivative of the Scaled Bézier Function

As far as Bernstein polynomials are concerned, we get the partial differential form Euqation 25. We derive from this that, for $s = \lfloor t\rfloor$, we have

$$\frac{\partial}{\partial t_a}B_{I,S,R,L}(T) = \frac{1}{l_a}\frac{\partial}{\partial t_a}B_{I,R}(\frac{T}{L} - S) = \frac{r_a}{l_a}\left(B_{I^{(a,-1)},S,R^{(a,-1)},L}(T) - B_{I,S,R^{(a,-1)},L}(T)\right) \quad (26)$$

so that

Proposition 5.6 (Differentiation and Finite Differences of Sequences) *For* $0 \leq R = (r_a)_{a=1,\dots,d}$*, the function* $\mathcal{B}^{(0)}_{L,R}(\Gamma)$ *is* C^{R-1} *on* $\mathcal{M}$ *and we have:*

$$\frac{\partial}{\partial t_a}\left(\mathcal{B}^{(0)}_{l,r}(\Gamma)\right)(T) = \frac{r_a}{l_a}\left[\left(\mathcal{B}^{(0)}_{L,R^{(a,-1)}}(\tau^{(a,-l_a)}(\Gamma))\right)(T) - \left(\mathcal{B}^{(0)}_{L,R^{(a,-1)}}(\Gamma)\right)(T)\right]$$

Proof. First we prove the result for all $t \in \mathcal{M}\backslash\mathcal{Z}_d$, on which the curve $\mathcal{B}^{(0)}_{l,r}(\Gamma)$ is easily seen to be polynomial, hence infinitely differentiable.

$$\begin{array}{rcl}
\frac{\partial}{\partial t_a}\left(\mathcal{B}^{(0)}_{L,R}(\Gamma)\right)(T) & = & \sum_{S\in\mathcal{Z}_d}\sum_{0\leq I\leq R}\Gamma\left(L(S-I)\right)\frac{\partial}{\partial t_a}B_{I,S,L,R}(T) \\
& = & \sum_{S\in\mathcal{Z}_d}\sum_{0\leq I\leq R}\Gamma\left(L(S-I)\right)\frac{r_a}{l_a}\left(B_{I^{(a,-1)},S,R^{(a,-1)},L}(T) - B_{I,S,R^{(a,-1)},L}(T)\right) \\
& = & \sum_{S\in\mathcal{Z}_d}\sum_{0^{(a,-1)}\leq I\leq R^{(a,-1)}}\Gamma\left(L(S-I^{(a,1)})\right)\frac{r_a}{l_a}\left(B_{I,S,R^{(a,-1)},L}(T)\right) \\
& & -\sum_{S\in\mathcal{Z}_d}\sum_{0\leq I\leq R}\Gamma\left(L(S-I)\right)\frac{r_a}{l_a}\left(B_{I,S,R^{(a,-1)},L}(T)\right) \\
& = & \sum_{S\in\mathcal{Z}_d}\sum_{0\leq I\leq R^{(a,-1)}}\Gamma\left(L(S-I)-0^{(a,l_a)}\right)\frac{r_a}{l_a}\left(B_{I,S,R^{(a,-1)},L}(T)\right) \\
& & -\sum_{S\in\mathcal{Z}_d}\sum_{0\leq I\leq R}\Gamma\left(L(S-I)\right)\frac{r_a}{l_a}\left(B_{I,S,R^{(a,-1)},L}(T)\right) \\
& = & \frac{r_a}{l_a}\left[\left(\mathcal{B}^{(0)}_{L,R^{(a,-1)}}(\tau^{(a,-l_a)}(\Gamma))\right)(T) - \left(\mathcal{B}^{(0)}_{L,R^{(a,-1)}}(\Gamma)\right)(T)\right]
\end{array}$$

Now, for $R = 0$, we have $B^{(0)}_{L,R}(T) = \sum_{s\in\mathcal{Z}_d}\Gamma(-LS)\mathbb{1}_{[SL,(SL+L)[}(T)$. Consequently, $B^{(0)}_{L,1}(\Gamma)$ is C^0 (we remind the reader that the vector 1 is here considered as having *all* its coordinates equal to 1. The result follows by induction on $1 \leq R$. □

Proposition 5.7 (Differentiation and Finite Differences of Functions) *For* $0 \leq R = (r_a)_{a=1,\dots,d}$*, the curve* $\mathcal{B}^{(0)}_{L,R}(\Gamma)$ *is* C^R *on* $\mathcal{M}$ *and we have:*

$$\frac{\partial}{\partial t_a}\left(\mathcal{B}^{(0)}_{L,R}(\Gamma)\right)(T) = \frac{r_a}{l_a}\left[\left(\tau^{(a,l_a)}\left(\mathcal{B}^{(0)}_{L,R^{(a,-1)}}(\Gamma)\right)\right)(T) - \left(\mathcal{B}^{(0)}_{L,R^{(a,-1)}}(\Gamma)\right)(T)\right]$$

Definition 5.5 Let $\Phi : \mathbb{Z} \longrightarrow E$ be a sequence, or $\Phi : \mathbb{R} \longrightarrow E$ be a function. We define the *finite difference masks*:

- $\left(\Delta^{(a,l)}_{-}(\Phi)\right)(S) = \frac{1}{l}(\Phi(S) - \Phi(S^{(a,-l)}))$;
- $\left(\Delta^{(a,l)}_{+}(\Phi)\right)(S) = \frac{1}{l}(\Phi(S^{(a,l)}) - \Phi(S))$.

Notation 5.2 *For* $\omega \in \mathbb{N}$ *with* $0 \leq \omega \leq R$*, we denote by* $\mathcal{B}^{(\omega)}_{L,R}(\Gamma)$ *the function on* $\mathcal{M}$ *defined as the differential of order* ω *of* $\mathcal{B}^{(0)}_{L,R}(\Gamma)$*:*

$$\mathcal{B}^{(\omega)}_{L,R}(\Gamma) = \left(\mathcal{B}^{(0)}_{l,r-\omega}(\Gamma)\right)^{(\omega)}$$

Therefore, Proposition 5.6 and Proposition 5.7 can be restated as:

Proposition 5.8 *The first order partial derivatives of* $\mathcal{B}_{l,r}(\Gamma)$ *can be computed in two ways through finite differences:*

- *On the sequence by* $\frac{\partial}{\partial t_a}\left(\mathcal{B}_{L,R}(\Gamma)\right)(T) = -r_a\left(\mathcal{B}^{(0)}_{L,R^{(a,-1)}}(\Delta^{(a,l_a)}_{-}(\Gamma)\right)(T)$*;*

- *On the function by* $\frac{\partial}{\partial t_a}\left(\mathcal{B}_{L,R}(\Gamma)\right)(T) = r_a \Delta_+^{(a,l_a)}\left(\mathcal{B}^{(0)}_{L,R^{(a,-1)}}(\Gamma)\right)(T)$

The following immediately follows by induction:

Proposition 5.9 *for $R \geq 0$ and $\omega \in \mathbb{N}^d$ with $\omega \leq R$, we can compute the differential with order ω of $\mathcal{B}_{L,R}(\Gamma)$, by applying an $\omega-$differentiation mask either to the sequence Γ by*

$$\left(\mathcal{B}^{(\omega)}_{L,R}(\Gamma)\right) = \frac{R!}{(R-\omega)!}\mathcal{B}^{(0)}_{L,R\omega}\left(\left(-1_{\mathcal{M}'}\right)^{|\omega|}\Delta_-^{\omega}(\Gamma)\right)$$

or to the function $\mathcal{B}_{L,R}(\Gamma)$ itself by

$$\left(\mathcal{B}^{(\omega)}_{L,R}(\Gamma)\right) = \frac{R!}{(R-\omega)!}\left(\Delta_+^{\omega}\right)\left(\mathcal{B}^{(0)}_{L,R-\omega}(\Gamma)\right)$$

5.3 Bernstein Based Differential $B-$Splines Family

Definition 5.6 We consider, for $P \in \mathbb{N}^d$, for $I \in \mathcal{Z}_d$, for $S \in \mathcal{Z}_d$, for $R \in \mathbb{N}^d$, a function $B_{I,S,P,R,L} \in \mathcal{M}'^{\mathcal{M}}$, based on the function $B_{I,S,R,L}$ defined in Definition 5.3, by the following inductive definition:

- $B_{I,S,0,R,L} = \frac{1}{L^{[R]}} B_{I,S,R,L}$
- For $P \geq 0$ and $T = (t_a)_{a=1,\dots,d}$, we set:

$$D_{I,S,P^{(a,1)},R,L}(T) = (p_a+1)\int_{\infty}^{t_a} D_{I,S,P,R,L}(T^{(a,u-t_a)})du$$

The family of piecewise polynomial functions thus defined is called the *Bernstein-based differential $B-$spline family.*

Theorem 5.1 *Bernstein-based differential $B-$spline family is a differential $B-$spline family as defined through Definition 4.3.*

The proof follows directly

- from Definition 5.6 which yields the differential property;
- from Equation (26), which can be integrated, and generalized for all $P \in \mathbb{N}^d$ gives us the commutation with the finite differences property;
- from Remark 5.2 which gives us the shift property;
- and from Remark 5.1 which gives us the partition of unity property.

6 Questions

Some questions remain to be addressed before final submission. Here is a non exhaustive, not always very specific list of short term questions we could think of. Some of them are required to fully justify significant results present in the draft to their target level of generality.

6.1 Questions relating to integrals and differentials

Question 6.1 (Product of Analyzable spaces) *The finite product $\mathcal{M} = \prod_{a=1}^{d} \mathcal{A}_a$ of analyzable spaces over the same ring is an analyzable space for the product algebra structure, the product measure, and the lexicographic order.*

Question 6.2 (Fubini Property) *For $x_1, x_2 \in \mathcal{A}$ and $y_1, y_2 \in \mathcal{B}$, for some analyzable spaces $\mathcal{A}$ and $\mathcal{B}$. Let $\mathcal{M}'$ be another analyzable space. Assume a function $f : \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{M}'$ is integrable for the product measure on $\mathcal{A} \times \mathcal{B}$. The the following integrals exit and are equal:*

$$\int_{x_1}^{x_2} \left(\int_{y_1}^{y_2} f(x,y)dy \right) dx = \int_{y_1}^{y_2} \left(\int_{x_1}^{x_2} f(x,y)dx \right) dy$$

Moreover, both integrals are equal to the integral of f over the interval $[(x_1, y_1); (x_2, y_2)]$ in the product analyzable space $\mathcal{A}$ and $\mathcal{B}$.

Question 6.3 (Differentiation of an Integral with Parameter) *For Ω a measurable set in an analyzable space $\mathcal{A} \times \mathcal{B}$ and $f : \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{M}'$ a continuous function such that for all $x \in \mathcal{A}$ the partial function $y \longmapsto f(x,y)$ is differentiable on $\mathcal{B}$, and its differentiation $\frac{\partial}{\partial y} f(x,y)$ is continuous with respect to (x,y). Then the function $y \longmapsto \int_\Omega f(x,y)dx$ is differentiable and we have:*

$$\frac{\partial}{\partial y} \left(\int_\Omega f(x,y)dx \right) = \int_\Omega \left(\frac{\partial}{\partial y} f(x,y) \right) dx$$

Question 6.4 *Show that the derivative of a polynomial function in an analyzable space is the usual formula.*

Question 6.5 *The primitive of a characterisics function of an interval in an analyzable space is continuous.*

6.2 Questions on rapidly decreasing functions

Question 6.6 (see also Theorem 2.1) *A multi-sequence $\mathbf{u} = (u(I))_{I \in \mathcal{Z}_d} \in \mathcal{M}'^{\mathcal{Z}_d}$ is isotropic and rapidly decrasing if and only if, for $a = 1, \dots, d$, there exist one-dimensional rapidly decreasing functions $\mathbf{u}_a = (u_a(I))_{I \in \mathcal{Z}_1} \in \mathcal{M}'^{\mathcal{Z}_1}$ such that $\mathbf{u} = \bigotimes_{a=1}^{d} \mathbf{u}_a$.*

Question 6.7 *Generalize to the general Cartesian product of arbitrary analyzable spaces the result (see Lemma 2.1 and Lemma 2.2).*

Lemma 6.1 *Let* $\mathbf{u}$ *a rapidly decreasing multi-sequence and let* π *be an* $\mathcal{M}'-$*valued polynomial function on* $\mathcal{M}$. *For* $I \in \mathcal{Z}_{d-1}$ *and* $i \in \mathcal{A}_d$, *let us denote by* $u(I,i)$ *[resp.* $\pi(I,i)$*] the image under* $\mathbf{u}$ *[resp. under* π*] of the concatenation of* I *and* (i). *Then the multi-sequence defined on* $\mathcal{Z}_{d-1}$ *by*

$$s_d(I) = \sum_{i \in \mathcal{A}_d} |u(I,i)||\pi(I,i)|$$

is well defined and bounded on $\mathcal{Z}_{d-1}$.

Proof. It is sufficient to prove this property when π is a monomial and, due to Remark 2.1, it is sufficient to prove it for polynomials of degree 0. In other words, we just need to show that the sum of the values of the multi-sequence $\mathbf{u}$ itself is absolutely convergent.

First we prove that for $d \geq 1$, the sum:

$$s_d(I) = \sum_{i \in \mathcal{A}_d} u(I,i)$$

is well defined for $I \in \mathcal{Z}_{d-1}$, and that the multi-sequence $\mathbf{s_d}$ itself is rapidly decreasing on $\mathcal{Z}_{d-1}$. Since $\mathbf{u}$ is rapidly decreasing, we can find $K > 0_{\mathcal{M}'}$ such that for $I \in \mathcal{Z}_{d-1}$ and $i \in \mathcal{A}_d$, we have $u(I,i) \leq K$ and $i^2 u(I,i) \leq K$. For $N \in \mathbb{N}^*$, we have

$$\begin{array}{rcl} (N!)^2 \sum_{i=1}^{N} |u(I,i)| & \leq & (N!)^2 |u(I,1)| + \sum_{i=2}^{N} i^2 |u(I,i)| * \frac{(N!)^2 1_{\mathcal{M}'}}{i^2} \\ & \leq & (N!)\,(K + K * 2) \end{array}$$

Note that the expression $\frac{N!}{i^2} 1_{\mathcal{M}'}$ denotes a well defined element of the algebra $\mathcal{M}'$ over the ring $\mathcal{R}$. Indeed, by expanding the expression of $(N!)^2$ and simplifying by i^2 to get an integer value, which is then multiplied by $1_{\mathcal{M}'}$ in the algebra $\mathcal{M}'$.

Hence, we get $\sum_{i \in \mathbb{N}^*} |u(I,i)|$ is well defined and bounded by $3K$. By a similar argument for $i < 0$, we get that $\sum_{i \in \mathcal{A}_d} |u(I,i)|$ is well defined and bounded on $\mathcal{Z}_{d-1}$. □

Lemma 6.2 *Let us consider the multi-sequence* $\mathbf{v}$ *defined on* $\mathcal{Z}_{d-1}$ *by* $v(I) = \sum_{i \in \mathcal{A}_d} u(I,i)$, *which is well-defined due to from Lemma 2.1, Then,* $\mathbf{v}$ *is a rapidly decreasing multi-sequence.*

Proof. Let π be a polynomial function on $\mathcal{Z}_{d-1}$. Then, by considering π as a function on $\mathcal{Z}_d$ (which does not depend on the d^{th} coordinate), we get by Remark 2.1 that the multi-sequence $I \longmapsto \pi(I)u(I)$ is rapidly decreasing. From Lemma 2.1, we get that the multi-sequence $I \longmapsto \pi(I)v(I)$ on $\mathcal{Z}_{d-1}$ is bounded, which proves that $\mathbf{v}$ is rapidly decreasing. □

6.3 differentiation of a product

Question 6.8 *Let* $\mathbf{u} = (u(I))_{i \in \mathcal{Z}_d}$ *be an* ω*-differentiation mask with* $\sum_{i=1}^{d} \omega_a = 1$. *and* $\mathbf{v}^{(1)} = (v^{(1)}(I))_{i \in \mathcal{Z}_d}$ *be a polynomial function and* $\mathbf{v}^{(2)} = (v^{(2)}(I))_{i \in \mathcal{Z}_d}$ *be a moderately increasing multi-sequence in* ${\mathcal{M}'}^{\mathcal{Z}_d}$. *We denote as usual* $(\mathbf{v^{(1)}v^{(2)}})(I) = v^{(1)}(I)v^{(2)}(I)$. *Then,* $\mathbf{v^{(1)}v^{(2)}}$ *is moderately increasing, and we have:*

$$\Delta_{\mathbf{u}}(\mathbf{v^{(1)}v^{(2)}}) = \mathbf{v^{(1)}}\Delta_{\mathbf{u}}(\mathbf{v^{(2)}}) + \mathbf{v^{(2)}}\Delta_{\mathbf{u}}(\mathbf{v^{(1)}})$$

Proposition 6.1 *Let $\mathbf{\Delta}_a$ be a differentiation operator over $\mathcal{A}_a$. We consider the partial differentials as defined in Definition 4.2 using $\mathbf{\Delta}_a$. Let $\mathbf{v}^{(1)} = (v^{(1)}(I))_{i\in\mathcal{Z}_d}$ be a polynomial function and $\mathbf{v}^{(2)} = (v^{(2)}(I))_{i\in\mathcal{Z}_d}$ be a moderately increasing multi-sequence in ${\mathcal{M}'}^{\mathcal{Z}_d}$. We denote as usual $(\mathbf{v^{(1)}v^{(2)}})(I) = v^{(1)}(I)v^{(2)}(I)$. Also denoting $T = (t_a)_{a=1,\dots,d}$, for s_1 and s_2 in $\mathcal{A}_a$, also denoting dt the measure underlying the analyzable space $\mathcal{A}_a$, we have:*

$$\begin{aligned}\int_{s_1}^{s_2} \mathbf{v^{(1)}}(T^{(a,t-t_a)}\frac{\partial}{\partial t_a}(\mathbf{v^{(2)}})(T^{(a,t-t_a)}dt &= \left[\mathbf{v^{(1)}}(T^{(a,t-t_a)}\mathbf{v^{(2)}}(T^{(a,t-t_a)}\right]_{t=s_1}^{t=s_2} \\ &- \int_{s_1}^{s_2} \mathbf{v^{(2)}}(T^{(a,t-t_a)}\frac{\partial}{\partial t_a}(\mathbf{v^{(1)}})(T^{(a,t-t_a)}dt\end{aligned}$$

Corollary 6.1 *Under the assumptions and notations of Proposition 6.1, Then, $\mathbf{v^{(1)}v^{(2)}}$ is moderately increasing, and we have:*

$$\frac{\partial}{\partial t_a}(\mathbf{v^{(1)}v^{(2)}}) = \mathbf{v^{(1)}}\frac{\partial}{\partial t_a}(\mathbf{v^{(2)}}) + \mathbf{v^{(2)}}\frac{\partial}{\partial t_a}(\mathbf{v^{(1)}})$$

6.4 Questions for error models and analysis

Question 6.9 *Define h^α for h in an analyzable space (or cartesian product) of a analyzable spaces (e.g. through power series) and $\alpha \in \mathbb{R}_+^*$.*

Question 6.10 *Define random variables with values in analyzable spaces and their standard deviations.*

6.5 Questions Relative to Convergence and Estimation

Definition 6.1 [Convergence Order of a Differentiation Mask] Let $\mathbf{u}$ be an $\omega-$differentiation mask (Definition 2.4). The *convergence order* of $\mathbf{u}$ is the greatest integer $\rho \in \mathbb{N}^d$ with $\rho \geq max_{a=1,\dots,d}(\omega_a)$ such that for all integer vector $k \neq \omega$ such that $\omega_a \leq k_a \leq \rho$ for $a = 1, \dots, d$, we have

$$\sum_{I\in\mathcal{Z}_d}\left(\prod_{a=1}^{d}(I(a)^{k_a})\right)u(I) = 0 \tag{27}$$

We shall use the convention that the convergence order is zero if there is no integer ρ satisfying Equation (27) for all integer vector $k \neq \omega$ such that $\omega_a \leq k_a \leq \rho$ for $a = 1, \dots, d$.

For the sake of clarity, we shall denote by $\Delta_{\mathbf{u}}^\omega$ the associated digital differentiation operator associated to $\mathbf{u}$ when $\mathbf{u}$ is a digital ω-derivative mask. Intuitively, and as appears in the proofs below, the order ρ of an $\omega-$differentiation operator is the maximal order of the coefficients of the Taylor development of the function $\frac{1}{h'^{\omega-1}}\Delta_{\mathbf{u}}^\omega(f) - f^{(\omega)}$ (strictly) below which all the coefficients vanish. As shown below, the convergence order ρ will determine the speed of convergence of $\frac{1}{h^{\omega-1}}\Delta_{\mathbf{u}}^\omega(f)$ to $f^{(\omega)}$, which is to distinguish from the differentiation order ω.

Question 6.11 *Can we improve the convergence results in Lemma 3.1, and consequently in Theorem 3.2 and Theorem 3.3, in case the convergence order in Definition 6.1 is greater than $\omega + 1$.*

Question 6.12 *Add the following to Theorem 3.2, possibly also improved by Question 6.11: if f is only C^ω and L is such that $\lim_{h \to 0} h.L(h) = 0$ and $\lim_{h \to 0} \frac{h^\alpha}{((h.L(h))^\omega} = 0$, then we have:*

$$\lim_{h \to 0} \left(\left(\frac{1}{(h')^{[\omega-1]}} \Delta_{\mathbf{u}_{L(h)}} \star \Gamma \right) (N) - f^{(\omega)}(Nh) \right) = 0_{\mathcal{M}'}$$

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